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The numerical solution of Richards' equation by means of method of lines and ensemble Kalman filter

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Abstract

Here a numerical technique based on the method of lines (MoL) for solving Richards' equation is presented. The Richards' equation deals with modeling infiltration of water into the unsaturated zone. By means of any kind of observations, some values of the state variable are assumed to be available at certain time points, in order to "correct" the numerical solution in the light of these observations. This is done by means of ensemble Kalman filter (EnKF), that is a data assimilation technique based on a Monte Carlo approach.

Advantages of this approach are discussed, in the light of existing bibliography.

Keywords: Unsaturated water flow; Richards' equation; Method of lines; Data assimilation; Ensemble Kalman filter

1. Introduction to the physical problem

The unsaturated (or vadose) zone is a buffer zone between land surface and aquifers: it can be considered a controlling agent in the transmission of water and contaminants. Here we will consider the vadose zone as a porous medium. "Water is held in an unsaturated medium by forces whose effect is expressed in terms of the energy state of the water, force being the negative gradient of energy. For an incompressible bulk material like water, the energy per unit volume can equivalently be considered as a pressure. The matric potential, or pressure, arises from interaction of water with a rigid matrix. Sometimes this quantity is called *capillary potential*, or *pressure*, or also *suction* (the negative of matric pressure)" [26].

The most basic measure of the water is volumetric *water content*, symbolized by θ, defined as the volume of water per bulk volume of the medium. This quantity is related with the pressure head by means of an algebraic equation, due to Van Genutchen, described in the following. Let us observe that water content values have to satisfy these constraints: $\theta_r \leq \theta \leq \theta_S$, where θ_S is the saturated water content (i.e. all the void spaces are full of water), whereas θ_r is the residual water content (i.e. the minimum possible water content, normally very close to zero).

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Modeling the infiltration of water (or any similar fluid) into unsaturated soils is really a hard task; the task is even harder if we want to model the infiltration into rocks, where the measures are more difficult and the numerical modeling is more necessary (see [7]). The dynamics of the infiltration process is driven essentially by gravity: capillary forces are generally negligible.

Assuming that the unsaturated soil is a homogeneous isotropic porous medium (and is a pretty strong assumption, indeed), the classical mathematical model for describing this process is the Richards' equation, that is a parabolic PDE, very *stiff* to integrate numerically.

For providing real-life models, the first, non straightforward problem is to set both initial and boundary conditions, and to settle the hydraulic parameters. Typically, these choices require an accurate knowledge of geology. On the other hand, there is a rich literature on elegant techniques that use data assimilation approach to improve the choice of parameters (see, for example, [14,6,25,9]).

We are going to focus on the infiltration process in one spatial dimension, that is of course the vertical one. For many applications, this hypothesis is sufficient to describe the infiltration. Moreover, even in presence of small soil heterogeneities, it has been proved that the effects of lateral movements can generally be neglected (see [28]). There are different equivalent formulations of Richards' equation, that differ on the state variable under consideration, that could be the *volumetric water content*, θ, or the *pressure-head*, ψ, typically correlated by some algebraic formulas (see [32,17,5]); here we will follow the notation in [17]:

$$\theta(\psi) = \frac{\alpha\,(\theta_S - \theta_r)}{\alpha + |\psi|^{\beta_2}} + \theta_r, \tag{1}$$

where α, β_2 are parameters depending on the medium.

Let us first introduce, in the one-dimensional notation, different formulations of Richards' equation:

$$C(\psi)\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial z}\left[K(\psi)\left(\frac{\partial\psi}{\partial z} - 1\right)\right] \quad \psi\text{-based form} \tag{2a}$$

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\left(\frac{\partial\theta}{\partial z} - 1\right)\right] \quad \theta\text{-based form} \tag{2b}$$

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[K(h)\left(\frac{\partial h}{\partial z} - 1\right)\right] \quad \text{mixed form.} \tag{2c}$$

In these formulas, the real valued functions $K(\psi)$ and $C(\psi)$ represent respectively *hydraulic conductivity* and *hydraulic capacity*. $K(\psi)$ describes the ease with which the water can move through pore spaces. It depends on the intrinsic permeability of the material and on the degree of saturation, and on the density and viscosity of the fluid. Hydraulic conductivity in saturated conditions is denoted by K_S. Here, following the notation in [17], adopted also in [18,30], $K(\psi)$ and $C(\psi)$ can be defined as

$$K(\psi) = K_S\frac{A}{A + |\psi|^{\beta_1}}, \tag{3a}$$

$$C(\psi) = \frac{d\theta}{d\psi}, \quad \text{where } \theta \text{ is defined as in (1).} \tag{3b}$$

For the existence of analytical solution of Richards' equation, see [15], whereas for integrating it, in the wake of a rich literature (see, for example, [8,27,30,28]), we will follow the notation (2a).

2. The numerical method

2.1. Numerical integration: motivation of MoL

Parabolic PDE (2) is an advection–diffusion equation. In [15] an interesting survey of pure mathematical analysis of Richards' equation is provided; in particular, it is stressed that Richards' equation can be expressed as a general advection–diffusion equation, in the following way:

$$u_t = (a(u))_{xx} + (b(u))_x\,, \tag{4}$$