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On the approximation by product rules of weakly singular double integrals over the square

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Abstract

Double integrals with algebraic and/or logarithmic singularities are of interest in the application of boundary element method, e.g. linear theory of the aerodynamics of slender bodies of revolution and in many other fields, for example computational electromagnetics. Therefore, the numerical evaluation of such type of integrals deserves attention. In this connection we propose here product interpolatory rules based on suitable Jacobi zeros, giving numerical tests to show the goodness of the proposed algorithm as well as from a point of view of convergence also for the simplicity of implementation.

Keywords: Double integrals; Weakly singular kernel; Product quadrature rules

1. Introduction

Many important partial differential equation problems in homogeneous media, such as those of acoustic or electromagnetic wave propagation, can be represented in the form of integral equations on the boundary of the domain of interest. In order to solve these problems, the boundary element method (BEM) can be applied [3]. The advantage compared to discretization domain methods, e.g. finite element methods (FEM), is that only a discretization of the boundary is necessary, which significantly reduces the number of unknowns. However, a lot of work has been done in the recent years addressed to propose numerical schemes for evaluating integrals in the method of moments (MoM). We recall among others [11–14] and the literature cited therein. Integrals of the form

$$\int_{-1}^{1}\int_{-1}^{1} k(|x-y|)f(x,y)dxdy, \tag{1.1}$$

where the kernel k has a weakly algebraic and/or a logarithmic singularity and f is a smooth function, arise in many applications of the boundary element method, especially when the potential or flux is required near a boundary. This

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circumstance occurs in the study of contact problems [1], displacement around open crack tips [6], thin structures [9] and sensitivity problems [16]. Further, such kind of double integrals is also widely used in the linear theory of the aerodynamics of slender bodies of revolution [2,7]. The large number of applications justify the interest for the numerical evaluation of (1.1).

In order to approximate (1.1), we propose to evaluate the inner integral

$$\int_{-1}^{1} k(|x-y|)f(x,y)dx, \quad y \in (-1,1),$$

by a suitable product rule, and the second one by a Gaussian rule. Both rules can be generated numerically by making use, among other things, of appropriate modified moments [5,10]. The paper is organized as follows. In Section 2 we reason on the numerical evaluation of the integral (1.1) by introducing a suitable algorithm, while the related computational aspects are discussed in Section 3. In Section 4 we provide examples strictly related to applications in a classical problem in aerodynamics. Finally, in concluding Section 5 we summarize the findings of the present research.

2. Product quadrature rules

In what follows we consider double integrals of the form (1.1) where the kernel k can have both algebraic and logarithmic singularities, i.e.

$$k(|x-y|) = (1-x)^{\alpha}(1+x)^{\beta}|x-y|^{\lambda}\left(\log^{\eta}|x-y|\right)u(y), \quad \alpha,\beta,\lambda > -1,\ \eta \in \{0,1\},$$

with u a nonnegative and integrable function on $(-1,1)$.

Let $\{p_m(w)\}_{m=0}^{\infty}$ be the sequence of the orthogonal Jacobi polynomials associated with the weight $w(x) = w^{\gamma,\delta}(x) = (1-x)^{\gamma}(1+x)^{\delta}$, $\gamma,\delta > -1$.

Denoting by $\lambda_{m,k}$, $k=1,2,\ldots,m$, the Christoffel constants with respect to the weight w, the Lagrange polynomial $\mathcal{L}_m(f)$ interpolating the function $f(\cdot,y)$ at the zeros $x_{m,k}$, $k=1,2,\ldots,m$, of $p_m(w)$ can be written as

$$\mathcal{L}_m(f;x,y) = \sum_{k=1}^{m} \ell_{m,k}(w;x)f(x_{m,k},y),$$

where

$$\ell_{m,k}(w;x) = \sum_{i=0}^{m-1} a_{k,i}p_i(w;x), \quad k=1,2,\ldots,m,$$

with

$$a_{k,i} = \lambda_{m,k}p_i(w;x_{m,k}), \quad k=1,2,\ldots,m,\ i=0,1,\ldots,m-1,$$

are the fundamental Lagrange polynomials of degree $m-1$.

Setting

$$d_i(w;y) = \sum_{k=1}^{m} \lambda_{m,k}p_i(w;x_{m,k})f(x_{m,k},y), \quad i=0,1,\ldots,\ y \in (-1,1),$$

we can write

$$\mathcal{L}_m(f;x,y) = \sum_{i=0}^{m-1} d_i(w;y)p_i(w;x).$$

In order to compute the inner integral in (1.1) we consider the interpolatory product rule

$$\int_{-1}^{1} k(|x-y|)f(x,y)dx \approx \int_{-1}^{1} \mathcal{L}_m(f;x,y)k(|x-y|)dx$$
$$= \sum_{i=0}^{m-1} d_i(w;y)q_i(w;y), \quad y \in (-1,1),$$