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Some issues on the automatic computation of plane envelopes in interactive environments

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Abstract

This paper addresses some concerns, and describes some proposals, on the ellusive concept of envelope of an algebraic family of varieties, and on its automatic computation.

We describe how to use the recently developed Gröbner Cover algorithm to study envelopes of families of algebraic curves, and we give a protocol towards its implementation in dynamic geometry environments. The proposal is illustrated through some examples. A beta version of GeoGebra is used to highlight new envelope abilities in interactive environments, and limitations of our approach are discussed, since the computations are performed in an algebraically closed field.

Keywords: Envelope; Dynamic Geometry; Automatic computation; GröbnerCover algorithm

1. Introduction

This paper addresses some concerns, and describes some proposals, on the ellusive concept of envelope of an algebraic family of varieties, and on its automatic computation.

We will deal with these issues by restricting our framework to

- families of algebraic plane curves;
- in the context of dynamic geometry software (DGS).

Yet, as we will show below (see examples in this Section), even in this restricted setting we will need to reflect about the very basic concept of envelope, in order to be able to propose some sound algorithmic protocols for its computation.

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1.1. The manifold concept of envelope

It is clear that the idea of envelope of a familiy of curves is an elementary differential geometry notion. Despite this elementary character, it is a definition that does not seem to generate a unanimous consensus on its basic terms. For instance, let us consider a familiy of curves $\{C_\alpha : F(x, y, \alpha) = 0\}$, where for each value α of the parameter we assume $F(x, y, \alpha) = 0$ to be the implicit polynomial equation of the curve C_α. In Wikipedia,[1] the envelope of this familiy of curves is introduced as a certain curve which is tangent to each one of the C_α's in the family. But, immediately after, four definitions are proposed and discussed concerning the very same concept:

1. the envelope as the set of all points (x, y) such that there is an α verifying $\{F(x, y, \alpha) = 0, \frac{\partial F(x,y,\alpha)}{\partial \alpha} = 0\}$, i.e., as the projection over the (x, y)-plane of the points, in the (x, y, α)-3 dimensional space, belonging to the surface $F(x, y, \alpha) = 0$ and having tangent plane parallel to the α-axis (or being singular points and, thus, not having tangent plane, properly speaking),
2. the envelope as the set of limit points of intersections of nearby curves C_α,
3. (as previously introduced) the envelope as a curve tangent to all the given curves,
4. the envelope as the curve that bounds the planar region described by the points belonging to the curves in the family.

Finally, the Wikipedia points out that these four definitions are not, in general, coincident; and that they yield to different envelope sets, E_i, for each definition $i = 1, 2, 3, 4$. But this situation is, by no means, a problem with the Wikipedia only. In a previous paper [4] we have reviewed different reputed sources, some classical and some very modern, all of them expressing the existence of a plurality of approaches to the concept of envelope, as well as describing the many subtle and difficult aspects involved in handling this apparently elementary notion. We also have summarily described in the same paper [4] the limitations for computing a specific envelope with general purpose and well-known DGS, such as Sketchpad, Cabri, Cinderella or GeoGebra. See Section 2 for a discussion of the state of the art. A rough and immediate conclusion from that survey is the need for improvement in different directions:

- extending the computation of envelopes to families of curves of much more general type (i.e. not limited to lines or circles, etc.);
- identifying the resulting envelope curve as an element of DGS (so that the system is able to do further operations with this curve) and not as a mere graphical display;
- improving the reliability and accuracy of the envelope computation (currently merely conjectural in most DGS, often erroneously including some extra components or omitting some true components with respect to the correct envelope).

1.2. On some difficulties when computing envelopes

As mentioned above, some of the main difficulties dealing with envelopes do not dissapear even if we restrict ourselves to working in the frame of computing with families of planar curves build up by manipulating with DGS. Thus, in order to justify the use in this context of very powerful tools from computational algebraic geometry (such as Gröbner Cover, see Section 3), let us start by discussing the following simple examples, which can provide the reader an idea of the involved subtleties.

Example 1. The first example is about computing the envelope of a very simple family of lines in the (x, y)-plane. Here we consider the family of all lines parallel to the x-axis, say, described by $y = t$, for a single parameter t. Then its envelope is—according to one of the classical definitions, but choosing any of them will yield here the same output—the result of eliminating t from $y - t$, $\partial(y - t)/\partial t$. Since $\partial(y - t)/\partial t$ is -1, the system has no solution. There is no envelope.

Now let us make the following consideration. It is easy to imagine that someone working with a DGS could be making a construction, attempting to build up a certain family of curves, but – unfortunately – it could happen that the performed construction it is not necessarily the simplest possible one yielding the same family.

[1] https://en.wikipedia.org/wiki/Envelope_(mathematics).