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Algorithmic method to obtain combinatorial structures associated with Leibniz algebras

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Abstract

In this paper, we introduce an algorithmic process to associate Leibniz algebras with combinatorial structures. More concretely, we have designed an algorithm to automatize this method and to obtain the restrictions over the structure coefficients for the law of the Leibniz algebra and so determine its associated combinatorial structure. This algorithm has been implemented with the symbolic computation package Maple. Moreover, we also present another algorithm (and its implementation) to draw the combinatorial structure associated with a given Leibniz algebra, when such a structure is a (pseudo)digraph. As application of these algorithms, we have studied what (pseudo)digraphs are associated with low-dimensional Leibniz algebras by determination of the restrictions over edge weights (i.e. structure coefficients) in the corresponding combinatorial structures.

Keywords: Pseudodigraph; Combinatorial structure; Leibniz algebra; Structure theory; Algorithm

1. Introduction

Nowadays, the discovery of new links and relations between different fields of Mathematics has become one of the most stimulating research. In this sense, alternative techniques can be obtained to solve open problems, improving already-known theories or disclosing new ones. In this paper, we present a procedure which interconnects Graph Theory and Leibniz algebras. Our main goal is the following: we want to generalize the research started in [3] for Lie algebras to the case of Leibniz algebras. In the above-mentioned reference, a mapping between Lie algebras and combinatorial structures was introduced in order to translate properties of Lie algebras into the language of Graph Theory and vice versa. In this work, we introduce the analogous mapping between Leibniz algebras and combinatorial structures, providing us with a generalized translator for properties between these two objects. In this mapping, some relevant differences must be considered; as, for example, the appearance of loops and the assignation of vectors as edge weights in the combinatorial structure.

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J.-L. Loday [14] introduced Leibniz algebras at the 1990s as a particular case of non-associative algebras, providing a non-commutative generalization of Lie algebras. In this way, Leibniz algebras inherit an important key property of Lie algebras: the right-multiplication operator on an element of a Leibniz algebra is a derivation (indeed an inner derivation). In fact, many well-known results on Lie algebras can be extended to Leibniz algebras. Research on Leibniz algebras is large in the literature due to its theoretical relevance and its application to several fields of Applied Sciences. This is due to all the existing relations between associative algebras and other fields like Differential Geometry, Game Theory and Functional Analysis. However, many general questions about these algebras are still unanswered by using traditional techniques. An example of this fact is the still-unsolved classification of Lie and Leibniz algebras. According to Levi's and Malcev's theorems (see [13,15], respectively) for Lie algebras and the analogue for Leibniz algebras (see [1]), the classification of these algebras only requires classifying semisimple and solvable algebras when considering fields of characteristic zero. Semisimple Lie algebras were classified by Killing and Cartan in 1890 over the complex number field and semisimple Leibniz algebras were studied in [11] and in some cases they can be decomposed into the direct sum of simple Lie ideals for fields of characteristic zero. With the current techniques, researchers cannot successfully face up to the classification problem for solvable Lie and Leibniz algebras. Therefore, new and different properties of these algebras would provide alternative ways to solve such problems. To achieve this purpose, mathematicians have dealt during the last decades with different links between solvable algebras and other fields.

In this paper, we present a new algorithmic method to associate a combinatorial structure with a Leibniz algebra. In addition, we also design and implement algorithmic procedures devoted to impose the Leibniz identities and obtain a list with all the (pseudo)digraphs associated with a given Leibniz algebra.

This paper is structured as follows: after reviewing some well-known results on Graph Theory and Leibniz algebras in Section 2, Section 3 is devoted to define the algorithmic method to associate combinatorial structures with Leibniz algebras. Next, Section 4 shows an algorithm to evaluate Leibniz identities and determine the restrictions over the structure constants, in order to return the list of allowed and forbidden configurations for combinatorial structures associated with Leibniz algebras. In addition, we also show an algorithm to draw these configurations when they are (pseudo)digraphs. All this goes with a brief computational study, showing that the complexity order of the procedures here presented is polynomial. Finally, Section 5 is devoted to apply the algorithms introduced in the previous section in order to obtain the list of (pseudo)digraphs associated with low-dimensional Leibniz algebras.

In our opinion, the tools introduced in this paper are very useful and helpful since we are introducing a novel relation between Leibniz algebras and simplicial complexes, which allows us to translate properties on the first into others on the second. Hence, this can raise a better understanding of Leibniz algebras, in the sense already achieved in [6,8] for Lie algebras. More concretely, as will be indicated in this paper, the techniques which are introduced here provide a little step forward in the classification of Leibniz algebras and the study of their properties by means of the translations of the problem to the language of combinatorial structures and its corresponding study via these objects. For the sake of example, we explain how the classification problem to obtain the list of isomorphism classes for Leibniz algebras can be reduced to determining the list of non-isomorphic combinatorial configurations (or even only considering (pseudo)digraphs); then detecting those being associated with Leibniz algebras; and finally, establishing which are the configurations associated with isomorphic Leibniz algebras.

2. Preliminaries

We show some preliminary concepts on Leibniz algebras, bearing in mind that the reader can consult [14] for a general overview. From here on, we only consider finite-dimensional Leibniz algebras over the complex number field $\mathbb{C}$.

Definition 1. A *Leibniz algebra* $\mathcal{L}$ over a field $\mathbb{K}$ is a vector space with inner bilinear composition law $[\cdot,\cdot]$, which satisfies the so-called Leibniz identity

$$[[X, Y], Z] - [[X, Z], Y] - [X, [Y, Z]] = 0, \quad \forall\, X, Y, Z \in \mathcal{L}.$$

From now on, we use the notation $L(X, Y, Z) = [[X, Y], Z] - [[X, Z], Y] - [X, [Y, Z]]$.

If, in addition, $[X, X] = 0$ holds for all $X \in \mathcal{L}$, then $\mathcal{L}$ is called *Lie algebra*. In this case, we have that $[X, Y] = -[Y, X]$ for all $X, Y \in \mathcal{L}$ and the Leibniz identity is equivalent to the Jacobi identity.