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Padovan-like sequences and Bell polynomials

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Abstract

We study a class of Padovan-like sequences that can be generated using special matrices of the third order. We show that terms of any sequence of this class can be expressed via Bell polynomials and their derivatives using as arguments terms of another such sequence with smaller indices. Computer algebra system (CAS) Mathematica was used for cumbersome calculations and hypothesis-testing.

Keywords: Integer sequences; Padovan sequence; Fibonacci sequence; Partial Bell polynomials; Padovan polynomials

1. Introduction

Integer sequences appear in many branches of science. The Fibonacci and Padovan sequences have long been used in purely theoretical problems of mathematics (such as in the famous solution of 10th Hilbert problem by Yu.V. Matiyasevich [10]) as well as in many applications (see. e.g. Wikipedia page on Fibonacci heap [7]), and purely technical disciplines, such as architecture [11]. Recently, a new and elegant result of Viswanath [12] related to the properties of random Fibonacci numbers, stimulated the interest of researchers to the properties of these sequences in terms of the behavior of stochastic dynamical systems [2,4]. On the other hand, continues the study of algebraic relations of these sequences with combinatorial polynomials (Fibonacci, Padovan, Chebyshev, Kravchuk) naturally associated with them. Fibonacci numbers are known for more than two thousand years, Padovan numbers are much younger—they were introduced only recently [9]. Below, we will study Padovan-like sequences that can be generated using special matrices of the third order. We will find expressions for terms of one sequence in terms of the other sequence via Bell polynomials (also known in the combinatorial theory as "partition polynomials").

So-called matrix method of generation of such sequences and their various generalizations is widely used in many of the problems mentioned above. As an example one can mention recent papers [3,14,15], in which this method was used to study various algebraic properties (generating function, generalized Binet formula, sums, etc.) of these sequences. In contrast to this papers, also using the (different) matrix method, we establish a purely algebraic

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(polynomial) relationship between terms of various sequences. This allows us to construct a polynomial algorithm that "recalculates" terms of any of the sequences of this family to another sequence of the same family using CAS.

CAS Mathematica was used for cumbersome calculations and hypothesis testing.

Everywhere in what follows all matrices have dimension 3×3 and are being denoted by uppercase letters except the matrix

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

that plays a special role in our considerations. We write $[X]_{ij}$ to denote the (i, j)-entry, $1 \le i, j \le 3$ in a matrix X.

The following formulas for matrix e and any matrices $X, Y, X_1, \ldots, X_s$ are obvious:

$$e^2 = \mathbf{0}; \qquad [Xe]_{22} = [X]_{21}; \qquad [e]_{11} = [Xe]_{11} = 0; \qquad [XeY]_{11} = [X]_{11} \cdot [Y]_{21}; \tag{1}$$

$$eX_1eX_2e \ldots eX_se = \left(\prod_{r=1}^{s} [X_r]_{21}\right) \cdot e; \qquad [eX_1eX_2e \ldots eX_se]_{11} = 0. \tag{2}$$

Furthermore, a (strong) composition of an integer n into k parts, is a way of writing n as the ordered sum of k strictly positive integers [1,6], but if zero terms are still allowed the composition is said to be weak. The numbers of these compositions (respectively) are:

$$Comp(n, k) = \binom{n-1}{k-1}, \qquad Compw(n, k) = \binom{n+k-1}{k-1} \tag{3}$$

and these numbers are set to be 0 for $n = 0$.

A partition of an integer n into k parts, is a way of writing n as (unordered) sum of k strictly positive integers [2]. It is easy to see that to any partition $1^{j_1}2^{j_2}3^{j_3} \ldots (n-k+1)^{j_{n-k+1}}$ of n into k parts, where $j_1 + j_2 + j_3 + \cdots + j_{n-k+1} = k$ and $1 \cdot j_1 + 2 \cdot j_2 + 3 \cdot j_3 + \cdots + (n-k+1) \cdot j_{n-k+1} = n$, there corresponds exactly $\frac{k!}{j_1!j_2!j_3!\ldots j_{n-k+1}!}$ compositions of n into k parts via ordering the summands in the partition. So, the sum of monomials $x_{i_1}x_{i_2} \ldots x_{i_k}$, where the sequences $\{i_1, i_2, \ldots, i_k\}$ are running over all compositions of n into k parts and where $\{x_r\}_{r \ge 1}$ is the (infinite) set of commuting variables, is equal to the analogous sum over all partitions of n into k parts:

$$\begin{aligned} \sum_{Comp} x_{i_1}x_{i_2} \ldots x_{i_k} &= \sum_{Part} \frac{k!}{j_1!j_2! \ldots j_{n-k+1}!} x_1^{j_1} x_2^{j_2} \ldots x_{n-k+1}^{j_{n-k+1}} \\ &= \frac{k!}{n!} B_{n,k}(x_1 \cdot 1!, x_2 \cdot 2!, \ldots, x_{n-k+1} \cdot (n-k+1)!) \\ &= B^*_{n,k}(x_1, x_2, \ldots, x_{n-k+1}), \end{aligned} \tag{4}$$

where

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{j_1!j_2! \ldots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \ldots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}} \tag{5}$$

are the partial Bell polynomials, and where the sum is running over all $(n-k+1)$ sets $\{j_1, j_2, \ldots, j_{n-k+1}\}$ of non-negative (and non-zero) integers such that $j_1 + j_2 + j_3 + \cdots + j_{n-k+1} = k$ and $1 \cdot j_1 + 2 \cdot j_2 + 3 \cdot j_3 + \cdots + (n-k+1) \cdot j_{n-k+1} = n$ (see [5]).

Mathematica 9 contains function $Bell Y\left[n, k, \{x_1, x_2, \ldots, x_{n-k+1}\}\right]$ which gives the partial Bell polynomial $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$, but here we use in fact a slightly modified form of these polynomials

$$B^*_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \frac{k!}{n!} B_{n,k}(x_1 \cdot 1!, x_2 \cdot 2!, \ldots, x_{n-k+1} \cdot (n-k+1)!),$$

according to formula (4).

Example 1. The full set of compositions of $n = 5$ into $k = 3$ parts is

$$\{\{3, 1, 1\}, \{2, 2, 1\}, \{2, 1, 2\}, \{1, 3, 1\}, \{1, 2, 2\}, \{1, 1, 3\}\}$$