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On estimation of the global error of numerical solution on canard-cycles

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Abstract

Under study is the behavior of the global error of numerical integration in the two-variable mathematical model of a heterogeneous catalytic reaction. Numerical estimation of the global error indicates that there is a high sensitive dependence of the solutions on initial conditions due to the existence of a tunnel-type bundle of trajectories which is formed by the stable and unstable canards. We show that the exponential growth of the norm of the fundamental matrix of solutions of the system linearized around a stable canard-cycle yields exponential growth of the leading term in the global error of numerical solution.

Keywords: Nonlinear dynamical system; Canards; Global error of numerical integration; Chemical kinetics

1. Introduction

We study numerically a kinetic model of a catalytic reaction described by a system of ordinary differential equations (ODEs). In classical strategies of nonlinear oscillations analysis the stress is on the individual solution curves and their properties. Here we are more concerned with families of such curves (bundles of trajectories) and hence with the global behavior of the flow in the phase space of the system. Therefore, a global error in long-term numerical integration of ODEs is of interest. We refer to the local expansion of orbits starting arbitrarily close together as the *sensitive dependence* on initial conditions.

In studies of some kinetic models of the form

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \mu\dot{x}_2 = f_2(x_1, x_2, z), \qquad \dot{z} = \varepsilon h(x_1, x_2, z) \tag{1}$$

with fast (x_2), intermediate (x_1), and slow (z) variables and small parameters μ and ε, the relaxation and mixed-mode oscillations and also chaotic dynamics were found [7,25,26]. Consider the one-parameter family of two-dimensional

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dynamical subsystems with the parameter z

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \mu\dot{x}_2 = f_2(x_1, x_2, z), \quad 0 \le z \le 1, \tag{2}$$

which is obtained from (1) in the limit as $\varepsilon \to 0$. It was observed that, for solutions of (1), high parametric sensitivity on initial conditions can appear in some regions of the phase space $\mathbb{R}^3$ and, in particular, in a neighborhood of the canard-cycles of system (2).

In connection therewith, we formulate in [7] the problem of finding a maximal family of canards in a system of type (2). Note that, the so-called *canard solutions* were discovered in slow–fast systems of such sort, which include segments of the trajectory that remain close to the unstable part of the *critical manifold* $f_2(x_1, x_2, z) = 0$ [1,15]. Of the greatest interest to us is the case when there is an interval I of the parameter z, where the two families of canard-cycles co-exist in (2): stable canards with head and unstable canards without head [7]. In this case, for every $z \in I$, stable and unstable canards form a bundle of trajectories of tunnel type whose trajectories have high parametric sensitivity on initial conditions.

Numerical study of the families of canards in the two-variable system (2) is a complex problem, especially when parameter μ is sufficiently small and we need to integrate precisely the orbitally stable canards that are close to the *maximal canard*. This is caused by the fact that the orbital stability is an integral characteristic, and a stable cycle can have a part of the phase path with the positive finite-time Lyapunov exponents that give the growth rate of small perturbations. In [12] we introduced a new approach to analyze the parametric sensitivity of a system of ODEs to initial conditions, which uses the finite-time Lyapunov stability analysis.

In [17] it is shown that under numerical integration of the orbitally stable cycles close to a canard without head the trajectories jump erratically left and right from the unstable part of the slow manifold rather than clearly track the stable canard. This illustrates that the numerical solution can be far from the exact solution of an ODE system, when the trajectory is in the phase space region where the bundle of trajectories of tunnel-type exists, even in the case when the local error is approximately 10^{-11}. This confirm that, controlling only the local error, we cannot determine the moment when the global error starts to grow.

Another difficulty arising under numerical integration of canard limit cycles is that the interval of parameter z values corresponding to the maximal family of canard-cycles is approximately

$$(z_c - M\mathbf{e}^{-\frac{1}{k\mu}}, z_c), \quad \text{or} \quad (z_c, z_c + M\mathbf{e}^{-\frac{1}{k\mu}}), \quad \text{or} \quad (z_c - M\mathbf{e}^{-\frac{1}{k\mu}}, z_c + M\mathbf{e}^{-\frac{1}{k\mu}}),$$

where z_c is such that the system has the maximal canard, and $M, k > 0$ are some constants. This phenomenon is referred to as a *canard explosion*. It occurs frequently in two-dimensional kinetic models of chemical reactions [2,7].

The behavior of the global error of numerical solution on canard-cycles combined with the fact that the canard-cycles exist in parameter ranges that are exponentially small in relation to the small parameter μ make these maximal families of canards computationally challenging. On the other hand, in [14,22] it is shown that in the case of a system of three ODE's with two slow variables the canards can exist when the parameter changes in some larger interval.

Let us note the importance of studying such sensitive objects as canards. First, for some real kinetic system described by (1), the degenerate system (2) include other parameters except z whose values are known only as some approximation. But "a canard's life is short", and under a rather small change in the input parameters canard may disappear. Therefore, an opinion may arise that a canard is only "a mathematical phenomenon" and is not so important for modeling. However, in [26] we showed that the mixed-mode oscillations and even chaos in the system (1) can be constructed if the knowledge of global dynamics of a one-parameter family of subsystems (2) is available. In this case, canards may be an essential singularity. In other words, the maximal family of periodic solutions may include canards also under perturbations of some other input parameters that are not so small as $\mathbf{e}^{-1/(k\mu)}$.

Trying to overcome the computational difficulties arising under numerical study of the maximal families of canard-cycles in the case when the small parameter in the system does not approach zero, in [9] we proposed an analytical approach to estimate the localization of locally invariant manifolds which are $O(\mu)$ close to the critical manifold and generate a bundle of trajectories of the tunnel type. This method uses an envelope of the straight lines passing through the points of contact of the isoclines $f_1/f_2 = \text{const}$ with different slopes and the vector field.

The global problem that we deal with in this paper is to develop criteria that allow us to find some subregions with high parametric sensitivity to initial conditions in the phase space of a dynamical system. For this purpose, we suggest to consider the *singular trajectories* (see [29] for definitions) that generate bundles of the trajectories (shower-