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Limit cycles bifurcating from a degenerate center

Jaume Llibre[a,*], Chara Pantazi[b]

[a] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C, 08193 Bellaterra, Barcelona, Catalonia, Spain
[b] Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, (EPSEB), Av. Doctor Marañón, 44–50, 08028 Barcelona, Spain

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Abstract

We study the maximum number of limit cycles that can bifurcate from a degenerate center of a cubic homogeneous polynomial differential system. Using the averaging method of second order and perturbing inside the class of all cubic polynomial differential systems we prove that at most three limit cycles can bifurcate from the degenerate center. As far as we know this is the first time that a complete study up to second order in the small parameter of the perturbation is done for studying the limit cycles which bifurcate from the periodic orbits surrounding a degenerate center (a center whose linear part is identically zero) having neither a Hamiltonian first integral nor a rational one. This study needs many computations, which have been verified with the help of the algebraic manipulator Maple.

Keywords: Polynomial differential systems; Centers; Limit cycles; Averaging theory

1. Introduction

Hilbert in [16] asked for the maximum number of limit cycles which real polynomial differential systems in the plane of a given degree can have. This is actually the well known *16th Hilbert Problem*, see for example the surveys [17,18] and references therein. Recall that a *limit cycle* of a planar polynomial differential system is a periodic orbit of the system isolated in the set of all periodic orbits of the system.

Poincaré in [22] was the first to introduce the notion of a center for a vector field defined on the real plane. So according to Poincaré a *center* is a singular point surrounded by a neighborhood filled of periodic orbits with the unique exception of the singular point.

Consider the polynomial differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

and as usually we denote by $\dot{} = d/dt$. Assume that system (1) has a center located at the origin. Then after a linear change of variables and a possible scaling of time system (1) can be written in one of the following forms

$$\text{(A)} \quad \begin{array}{l} \dot{x} = -y + F_1(x, y), \\ \dot{y} = x + F_2(x, y), \end{array} \qquad \text{(B)} \quad \begin{array}{l} \dot{x} = y + F_1(x, y), \\ \dot{y} = F_2(x, y), \end{array} \qquad \text{(C)} \quad \begin{array}{l} \dot{x} = F_1(x, y), \\ \dot{y} = F_2(x, y), \end{array}$$

* Corresponding author.
E-mail addresses: jllibre@mat.uab.cat (J. Llibre), chara.pantazi@upc.edu (C. Pantazi).

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with F_1 and F_2 polynomials without constant and linear terms. When system (1) can be written into the form (A) we say that the center is of *linear type*. When system (1) can take the form (B) the center is *nilpotent*, and when system (1) can be transformed into the form (C) the center is *degenerate*.

Due to the difficulty of this problem mathematicians have considered simpler versions. Thus Arnold [1] considered the *weakened 16th Hilbert Problem*, which consists in determining an upper bound for the number of limit cycles which can bifurcate from the periodic orbits of a polynomial Hamiltonian center when it is perturbed inside a class of polynomial differential systems, see for instance [9] and the hundred of references quoted therein. It is known that in a neighborhood of a center always there is a first integral, see [21]. When this first integral is not polynomial the computations become more difficult. Moreover, if the center is degenerate the computations become even harder.

In the literature we can basically find the following methods for studying the limit cycles that bifurcate from a center:

- The method that uses the Poincaré return map, like the articles [4,8].
- The one that uses the Abelian integrals or Melnikov integrals (note that for systems in the plane the two notions are equivalent), see for example section 5 of Chapter 6 of [2] and section 6 of Chapter 4 of [15].
- The one that uses the inverse integrating factor, see [11–13,25].
- The averaging theory [6,14,19,23,24].

The first two methods provide information about the number of limit cycles whereas the last two methods additionally give the shape of the bifurcated limit cycle up to any order in the perturbation parameter.

Almost all the papers studying how many limit cycles can bifurcate from the periodic orbits of a center, work with centers of linear type. There are very few papers studying this problem for nilpotent or degenerate centers. In fact, for degenerate centers as far as we known the bifurcation of limit cycles from the periodic orbits of a degenerate center only have been studying completely using formulas of first order in the small parameter of the perturbation. Here we will provide a complete study of this problem using formulas of second order, and as it occurs with the formulas of second order applied to linear centers that they provide in general more limit cycles than the formulas of first order, the same occurs for the formulas of second order applied to degenerate centers. Of course, the amount computations from first order to second order increases almost exponentially.

This paper deals with the weakened 16th Hilbert's problem but perturbing non-Hamiltonian degenerate centers using the technique of the averaging method of second order, see [14], and Section 2 for a summary of the results that we need here.

Since we want to study the perturbation of a degenerate center with averaging of second order, from the homogeneous centers the first ones that are degenerate, are the cubic homogeneous centers, see for instance [7]. In this class in [20] the authors studied the perturbation of the following cubic homogeneous center

$$\dot{x} = -y(3x^2 + y^2), \qquad \dot{y} = x(x^2 - y^2), \tag{2}$$

inside the class of all cubic polynomial differential systems, using averaging theory of first order. Here we study this problem but using averaging theory of second order.

System (2) has a global center at the origin (i.e. all the orbits contained in $\mathbb{R}^2 \setminus \{(0, 0)\}$ are periodic), and it admits the non-rational first integral

$$H(x, y) = (x^2 + y^2) \exp\left(-\frac{2x^2}{x^2 + y^2}\right).$$

The limit cycles bifurcating from the periodic orbits of the global center (2) have already been studied in the following two results, see [20] and [5], respectively.

Theorem 1. *We deal with differential system (2). Then the polynomial differential system*

$$\dot{x} = -y(3x^2 + y^2) + \varepsilon\left(\sum_{0\le i+j\le 3} a_{ij} x^i y^j\right),$$