Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 99 (2014) 125–138
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original article

Design and implementation of a multiscale mixed method based on a nonoverlapping domain decomposition procedure

A. Francisco [a], V. Ginting [b], F. Pereira [c,*], J. Rigelo [b]

[a] *Departamento de Engenharia Mecânica, Universidade Federal Fluminense, Av. dos Trabalhadores 420, Volta Redonda, RJ 27255-125, Brazil*
[b] *University of Wyoming, Department of Mathematics, 1000 East University Avenue, Laramie, WY 82071-3036, USA*
[c] *University of Wyoming, Department of Mathematics and School of Energy Resources, 1000 East University Avenue, Laramie, WY 82071-3036, USA*

Received 29 November 2011; received in revised form 26 August 2012; accepted 3 April 2013
Available online 4 September 2013

Abstract

We use a nonoverlapping iterative domain decomposition procedure based on the Robin interface condition to develop a new multiscale mixed method to compute the velocity field in heterogeneous porous media. Hybridized mixed finite elements are used for the spatial discretization of the equations. We define local, multiscale mixed basis functions to represent the discrete solutions in subdomains. Appropriate subspaces of the vector space spanned by these basis functions can be considered in the numerical approximations of heterogeneous porous media flow problems. The balance between numerical accuracy and numerical efficiency is determined by the choice of these subspaces. A detailed description of the numerical method is presented. Following that, numerical experiments are discussed to illustrate the important features of the new procedure and its comparison to the traditional fine grid simulations.

Keywords: Multiscale methods; Mixed finite elements; Domain decomposition

1. Introduction

We are concerned with the development of numerical procedures for the fast and accurate approximation of subsurface flows that can take advantage of heterogeneous (e.g., CPU–GPU clusters) processing units. Such units are relatively inexpensive and have larger computational power (by one or two orders of magnitude) than CPUs alone. In the case of second order elliptic equations, multiscale procedures have the potential to fit well within the above mentioned heterogeneous computational environments because GPUs can be efficiently used to solve many small local problems that are posed by such procedures.

We present a new multiscale method based on a nonoverlapping iterative domain decomposition for mixed finite elements introduced in [10]. Domain decomposition techniques for mixed methods distinct from the one used here can be found in, e.g., [5,8,12,13,15,16].

The development of multiscale numerical procedures for second order elliptic equations arising in porous media flow problems has attracted the attention of several research groups. We refer the reader to the recent publications

* Corresponding author.
E-mail address: felipepereira62@gmail.com (F. Pereira).

0378-4754/$36.00 © 2013 IMACS. Published by Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.matcom.2013.04.022

[7,11] and [17] for references and also a very comprehensive comparison of several procedures of this type. More recent developments include [19] along with [14] among many others. In [14] the authors make use of the concept of a multiscale flux basis. A similar concept will be used in this work.

In the work presented here we define an iterative domain decomposition algorithm taking advantage of multiple grids and a new family of local basis functions to approximate large problems efficiently.

The new procedure has some distinctive properties:

- Three length scales are introduced in the definition of the procedure: the solution is sought in the finest scale (h); multiscale basis functions are defined in the subdomains associated with the coarsest scale (H). Flux conservation is directly imposed on an intermediate scale ($\overline{H}$ with $h \le \overline{H} \le H$). If $\overline{H} = h$ the new method produces the mixed finite element solution in the finest scale.
- The approximation is locally conservative at the $\overline{H}$ scale through the use of mixed finite elements. Discontinuous coefficients and source terms can be naturally handled.
- A post-processing step is introduced to recover the solution in the fine grid, given the solution with flux conservation at the $\overline{H}$ scale. A downscaling procedure ensures local conservation at the h scale.
- The implementation of the iterative scheme, which is based on the Robin interface conditions imposed on the intermediate scale, is straightforward and converges fast.
- The procedure fits well in heterogeneous processing units (CPU–GPU): all the local problems can be efficiently solved in GPUs. These local problems are relatively small and fit in the GPU memory. Moreover, the iterative procedure associated with the global problem in a coarse grid takes place only at the boundaries of the coarse grid blocks, and can also run in GPUs.

This paper is organized as follows. In Section 2 we describe the mixed formulation of the elliptic problem considered in this work, next we give a step-by-step formulation of the new multiscale mixed method (MuMM), first introducing a discretization by hybridized mixed finite elements and a modified Robin condition, followed by the description of the MuMM iteration. In Section 3 the computational implementation is described: multiscale basis functions of Robin type are introduced and a post-processing step is discussed. Section 4 provides several numerical results for the new method. Finally, Section 5 contains our concluding remarks.

2. The multiscale mixed method (MuMM)

2.1. The pressure equation and its variational formulation

For simplicity in the presentation, we consider $\Omega \subset \mathfrak{R}^2$, a bounded domain with a Lipschitz boundary $\partial\Omega$. The authors are currently extending this work to problems in $\mathfrak{R}^3$. The pressure–velocity system in the single-phase porous media flows [6] is given by

$$\nabla \cdot \boldsymbol{u} = q, \quad \text{and} \quad \boldsymbol{u} = -K(\boldsymbol{x})\nabla p, \tag{2.1}$$

where $K(\boldsymbol{x})$ is known as the absolute permeability (a positive definite tensor), $\boldsymbol{u}$ is the Darcy velocity, p is the pressure and $q = q(\boldsymbol{x})$ represents sources and sinks. Typical boundary conditions that occur in subsurface flow problems are Dirichlet and Neumann, which are respectively expressed as

$$p = p_b \quad \text{on} \quad \Gamma_D, \qquad \boldsymbol{u} \cdot \boldsymbol{\nu} = u_b \quad \text{on} \quad \Gamma_N,$$

where $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and $\boldsymbol{\nu}$ is the outward unit normal vector.

Our numerical procedure is derived from the weak formulation of the above pressure–velocity system which is described globally. To do so, we first define the following spaces:

$$\begin{aligned} W(\Omega) &= L^2(\Omega), \\ H(\text{div}; \Omega) &= \{\boldsymbol{v} \in (L^2(\Omega))^2 \ \mid \ \text{div}\, \boldsymbol{v} \in L^2(\Omega)\}, \\ V_g(\Omega) &= \{\boldsymbol{v} \in H(\text{div}; \Omega) \ \mid \ \boldsymbol{v} \cdot \boldsymbol{\nu} = g \ \text{ on } \ \Gamma_N\}, \end{aligned}$$