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Time-splitting approximation of the Cauchy problem for a stochastic conservation law

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Abstract

In this paper, we present a time discretization of a first-order hyperbolic equation of nonlinear type set in $\mathbb{R}^d$ and perturbed by a multiplicative noise. Using an operator splitting method, we are able to show the existence of an approximate solution. Thanks to recent techniques of well-posedness theory on this kind of stochastic equations, we show the convergence of such an approximate solution towards the unique stochastic entropy solution of the problem, as the time step of the splitting scheme converges to zero.

Keywords: Stochastic PDE; Multiplicative noise; Itô integral; Time-splitting method; Young measures; Entropy solution

1. Introduction

We are interested in the Cauchy problem for a nonlinear hyperbolic scalar conservation law with a multiplicative stochastic perturbation of type:

$$\begin{cases} du + \operatorname{div} \vec{\mathbf{f}}(u)dt = h(u)dW & \text{in }]0,T[\times\mathbb{R}^d \times \Omega, \\ u(\omega,0,x) = u_0(x), & \omega \in \Omega,\ x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where div is the divergence operator with respect to the space variable (which belongs to $\mathbb{R}^d$), $d \geqslant 1$, $T > 0$ and $W = \{W_t, \mathcal{F}_t; 0 \le t \le T\}$ is a standard adapted one-dimensional continuous Brownian motion defined on the classical Wiener space $(\Omega, \mathcal{F}, P)$. By denoting $Q =]0,T[\times\mathbb{R}^d$, this equation has to be understood in the following way: P-a.s. in Ω and $\forall \varphi \in \mathcal{D}(Q)$

$$\int_Q u\partial_t\varphi + \vec{\mathbf{f}}(u).\nabla_x\varphi dxdt = \int_Q \int_0^t h(u)dW(s)\partial_t\varphi dxdt.$$

Note that, even in the deterministic case, a weak solution to a nonlinear scalar conservation law is not unique in general. The mathematical stake consists in introducing a selective criterion in order to identify the physical solution.

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In the present work we consider a stochastic version of the entropy condition proposed by S.N. Kruzhkov in the 70s, the one used in [2] and presented in Section 2.

We assume the following hypotheses:

H_1: $\vec{\mathbf{f}} : \mathbb{R} \to \mathbb{R}^d$ is a Lipschitz-continuous function with $\vec{\mathbf{f}}(0) = \mathbf{0}$.
H_2: $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function with $h(0) = 0$.
H_3: $u_0 \in L^2(\mathbb{R}^d)$.
H_4: There exists $M > 0$ such that $\operatorname{supp} h \subset [-M, M]$.
H_5: $u_0 \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$.[1]

Remark 1.
- H_1, H_2 and H_3 are claimed conditions from the theoretical point of view to ensure the well-posedness in the sense of [2]. Let us first mention that H_1 can be weakened by assuming that $\vec{\mathbf{f}}$ is a locally Lipschitz continuous function. Indeed, since the solution u is bounded by a constant M_1 depending only on M and $\|u_0\|_\infty$, the result holds by a truncation argument of $\vec{\mathbf{f}}$ outside $[-M_1, M_1]$. Secondly, since $\operatorname{div}[\vec{\mathbf{f}}(0)] = 0$, one can assume by convenience that $\vec{\mathbf{f}}(0) = \mathbf{0}$.
- H_4 and H_5 are specific conditions from the numerical analysis point of view. These are technical assumptions to control the estimates in the forthcoming lemmas, in particular to apply Lemma 3.3. Note that H_4 is a necessary condition to keep the solution u bounded.

1.1. Former results

Only few papers have been devoted to the theoretical study of hyperbolic scalar conservation laws with a multiplicative stochastic forcing: the development of a well-posedness theory has been done in [4–9] by the way of strong entropy solution, in [6] by the use of kinetic formulation, and in [2,3] with the notion of stochastic entropy solution. For a thorough exposition of all these papers, we refer the reader to the introduction of [1]. Concerning the numerical analysis of such stochastic problems, there is also, to our knowledge, few papers. Let us cite the work of Holden–Risebro [10] where a time-discretization of the equation is proposed by the use of an operator-splitting method. They proposed a result of convergence to prove the existence of pathwise weak solutions to the Cauchy problem for (1) set in $\mathbb{R}$. In the recent paper of Bauzet [1], a generalization of the work of Holden–Risebro [10] is proposed in a bounded domain D of $\mathbb{R}^d$. The author proved that the pathwise weak solution obtained in [10] is the unique entropy weak solution of the stochastic conservation law and that the whole sequence of approximation given by the time-splitting scheme converges strongly. Kröker–Rohde [11] are interested in a recent work in a method of handling the finite volume schemes for the approximate solution of the Cauchy problem for (1) and investigate on a space-discretization of the equation. For a class of strongly monotone numerical fluxes they established the pathwise convergence of a semi-discrete finite volume solution towards a stochastic entropy solution. Since the authors use a stochastic version of the compensated compactness approach, the study is restricted to the one-dimensional case.

1.2. Goal of the study and main result

In a recent published paper [1], a generalization of the time-splitting method introduced much earlier by Holden and Risebro [10] is proposed to approximate solution of the stochastic conservation law (1) set in a bounded domain D of $\mathbb{R}^d$ and with homogeneous Dirichlet boundary condition. Precisely, the author showed that the pathwise weak limit obtained in the former study of Holden and Risebro is the unique stochastic entropy solution of (1) and that the whole sequence of approximate solutions converges strongly with respect to all its variables. Our aim in the present paper is to prove that the tools developed in [1] are sufficiently strong to be extended to an unbounded domain and allow us to complete the work of Bauzet–Vallet–Wittbold [2] by a numerical analysis using their well-posedness theory for stochastic entropy solution. The main result of the present paper which deals with the convergence of our numerical scheme is stated in the following theorem.

[1] Where $BV(\mathbb{R}^d)$ denotes the set of integrable functions with bounded variation on $\mathbb{R}^d$.