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Filling holes with shape preserving conditions

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Abstract

Recently, several techniques have been developed to fill polygonal holes in a given surface by using $\mathcal{C}^1$-spline patches. Such techniques are based on the minimization of an energy functional which controls the fairness of the patch as well as its closeness to the original surface where it is known, that is, outside the hole. Nevertheless, the filling patch obtained tends to be flat due to the definition of the energy functional, so the used technique does not work properly in certain cases. Here we propose to generalize the filling method previously developed in other works in order to fill holes with some 'shape' conditions, i.e., in such a way that the filling patch 'inherits' as much as possible the shape of the original surface where it is known.

Keywords: Filling; Shape-preserving; Approximation; Powell–Sabin finite element; Minimal energy

1. Introduction

In the last few years, several variational methods have been developed in relation with the approximation or interpolation of a given set of scattered data. Most of these variational approaches consist of minimizing an energy functional that usually contains two terms: the first indicates how well the curve or surface approximates or interpolates the data set, while the second controls the degree of smoothness or fairness of the curve or surface. A wide range of minimization functionals have been proposed, derived from physical considerations (e.g. stretch energy or bending energy) or geometric entities (e.g. curve length, surface area or curvature). Discrete smoothing D^m-splines [1,2] provide specific examples of variational curves and surfaces. These splines minimize, in a finite element space, some quadratic functionals that contain terms associated with Sobolev seminorms. These variational methods receive considerable attention due to their efficiency and usefulness in the fitting and design of curves and surfaces.

On the other hand, the problem of filling holes or completing a 3D surface arises in all sorts of computational graphics areas, like CAGD, CAD–CAM, Earth Sciences, computer vision in robotics, image reconstruction from satellite and radar information, etc. Several works related to the field of filling holes have been published in the last few years (e.g., [7,9,12] and [5]).

Regarding the field of hole filling, the authors of this work have recently developed a technique to fill polygonal holes in a given surface by using $\mathcal{C}^1$-spline patches (see e.g. [5,8]). Such technique is based on the minimization of

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an energy functional which controls the fairness of the patch as well as its closeness to the original surface where it is known, that is, outside the hole. Nevertheless, the filling patch obtained tends to be flat due to the definition of the energy functional, in such a way that this approach is not adequate for certain applications.

In this work we propose to generalize the filling method previously developed in other works in order to fill holes with some *shape conditions*, i.e., in such a way that the filling patch 'inherits' as much as possible the shape of the original surface where it is known.

In real single-variable Calculus, there are definitions for increasing, decreasing, concave and convex functions. These definitions involve derivatives and make reference to shape characteristics of the functions. Following this idea, given a data function f with a hole, the method we propose in this work consists of 'estimating' the shape of f inside the hole by constructing functions that estimate the unknown derivatives of f inside the hole. To this end, we utilize the information available of f outside the hole. Then, we obtain a filling patch for f whose derivatives inside the hole be as close as possible to the 'estimated ones' of f, pretending in this way that the shapes of f and its filling patch be close.

This paper is organized as follows: in Section 2, we recall some preliminary concepts and we fix the notation. In Section 3 we formulate the problem we want to solve. Section 4 is devoted to the study of the problem: we prove the existence and uniqueness of solution and we show how to compute it. Finally, in Section 5 we give some graphical and numerical examples. In such examples we compare the results obtained when the filling patch is constructed with or without shape conditions. At the end of the paper, we briefly explain a method to obtain optimum values for the parameters involved in the considered functionals.

2. Notation and preliminaries

Let $D \subset \mathbb{R}^2$ be a polygonal domain (an open non-empty connected set) and let us consider the Sobolev space $\mathcal{H}^{r+1}(D)$, $r \geq 1$, whose elements are (classes of) functions u defined on D such that their partial derivatives (in the distribution sense) $\partial^\beta u$ belong to $L^2(D)$, with $\beta := (\beta_1, \beta_2) \in \mathbb{N}^2$ and $|\beta| := \beta_1 + \beta_2 \leq r + 1$. For any open subset $X \subset D$ we consider the usual inner semi-products

$$(u, v)_{m,X} := \sum_{|\beta|=m} \int_X \partial^\beta u(x) \partial^\beta v(x) dx, \quad m = 0, \ldots, r+1;$$

the seminorms

$$|u|_{m,X} := (u, u)_{m,X}^{1/2} = \left(\sum_{|\beta|=m} \int_X \partial^\beta u(x)^2 dx \right)^{1/2}, \quad m = 0, \ldots, r+1;$$

and the norm

$$\|u\|_X = \left(\sum_{m=0}^{r+1} |u|_{m,X}^2 \right)^{1/2} = \left(\sum_{|\beta| \leq r+1} \int_X \partial^\beta u(x)^2 dx \right)^{1/2}. \tag{1}$$

We will denote $\langle \cdot \rangle_n$ the usual Euclidean norm and $\langle \cdot\, ,\ \cdot \rangle_n$ the Euclidean inner product in $\mathbb{R}^n$.

Given $\alpha \geq 1$, let $\mathcal{T}$ be an α-triangulation of $\overline{D}$, i.e., a triangulation that satisfies the condition

$$1 \leq \frac{R_T}{2r_T} \leq \alpha$$

for all closed triangles $T \in \mathcal{T}$, R_T and r_T being the radii of the circumscribed and inscribed circles of T, respectively (see e.g. [13]). We consider the associated Powell–Sabin triangulation $\mathcal{T}_6$ of $\mathcal{T}$ (see e.g. [10]): The micro-triangles in $\mathcal{T}_6$ are obtained by joining the center Ω_T of the inscribed circle of each interior triangle $T \in \mathcal{T}$ to the vertices of T and to the centers $\Omega_{T'}$ of the inscribed circles of the neighboring triangles $T' \in \mathcal{T}$. When T has a side lying on the boundary of D, the point Ω_T is joined to the mid-point of this side, to the vertices of T and to the centers $\Omega_{T'}$ of the inscribed circles of the neighboring triangles $T' \in \mathcal{T}$. Hence, all the micro-triangles inside any $T \in \mathcal{T}$ have the incenter of T as a common vertex. Nevertheless, Powell–Sabin subtriangulation can be also obtained by using the barycenter instead of the incenter of T in the split procedure for certain triangulations (e.g. Δ^1-type triangulations).