Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 118 (2015) 293–301
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

Computer simulation of the interplay between fractal structures and surrounding heterogeneous multifractal distributions. Applications

Miguel Angel Martín[a,*], Miguel Reyes[b], Francisco Javier Taguas[a]

[a] *Department of Applied Mathematics, E.T.S.I. Agronomy, Avda. Complutense s/n, 28040 Madrid, Spain*
[b] *Department of Applied Mathematics, Computer Sciences Faculty, Campus de Montegancedo, 28660 Boadilla del Monte, Madrid, Spain*

Received 20 March 2014; received in revised form 30 October 2014; accepted 24 November 2014
Available online 11 December 2014

Abstract

In a large number of physical, biological and environmental processes interfaces with high irregular geometry appear separating media (phases) in which the heterogeneity of constituents is present. In this work the quantification of the interplay between irregular structures and surrounding heterogeneous distributions in the plane is made

For a geometric set $A \subset \mathbb{R}^2$ and a mass distribution (measure) μ supported in $S \subset \mathbb{R}^2$, being $A \subset S$, the mass $\mu(A(\varepsilon))$ gives account of the interplay between the geometric structure and the surrounding distribution. A computation method is developed for the estimation and corresponding scaling analysis of $\mu(A(\varepsilon))$, being A a fractal plane set of Minkowski dimension D and μ a multifractal measure produced by random multiplicative cascades. The method is applied to natural and mathematical fractal structures in order to study the influence of both, the irregularity of the geometric structure and the heterogeneity of the distribution, in the scaling of $\mu(A(\varepsilon))$. Applications to the analysis and modeling of interplay of phases in environmental scenarios are given.

Keywords: Fractals; Multifractals; Phases interplay; Fractal dimension; Entropy dimension

1. Introduction

In a large number of scenarios of different scientific fields, objects or regions with irregular borders appear surrounded by phases or media in which a high variability in the concentration of some constituents is a main ingredient. Such is the case of diffusion fronts in chemical reactions [9], wetting fronts in unsaturated soil and porous media [4,9], percolation boundaries [9], river basins [20], tumor growth [1], etc. The mass of such constituents near the border should influence the interplay between phases and thus its estimation becomes imperative. This ubiquitous situation where an irregular geometric structure appears surrounded by a highly heterogeneous distribution suggests it should be modelized and handled by Fractal Geometry tools. Above stimulating objective motivated this research.

* Corresponding author.
E-mail addresses: miguelangel.martin@upm.es (M.A. Martín), miguel.reyes@upm.es (M. Reyes), fj.taguas@upm.es (F.J. Taguas).

http://dx.doi.org/10.1016/j.matcom.2014.11.017
0378-4754/

After Mandelbrot [16], fractals have been widely used to quantify geometric irregularity in nature. Multifractality, a concept coming from the seminal ideas given in Mandelbrot [15], much more subtle and less popular indeed, has also been used to address heterogeneity. Thus, fractals and multifractals, have proven useful in describing and parameterizing the irregularity of shape via the fractal dimension and the complexity of spatial distributions via the multifractal spectrum, respectively. However, the use of both together to address the picture drawn above has only been addressed recently [18].

Let $A \subset \mathbb{R}^2$ be a fractal set and μ a mass distribution (measure) supported in $S \subset \mathbb{R}^2$ being $A \subset S$. However the amount of measure $\mu(A(\varepsilon))$ surrounding A within an "small neighborhood" $A(\varepsilon) = \{x \in S : \operatorname{dist}(A, x) < \varepsilon\}$ may be positive and, in such case, its relative amount respect to ε gives account of the geometric–measure interplay between both, the set A and the distribution μ.

For uniform measures the quantity $\mu(A(\varepsilon))$ is directly related with the geometric size of $A(\varepsilon)$, and consequently with the fractal dimension of A. For highly heterogeneous distributions however, as is the case of multifractal measures, where the mass concentration (local dimension) varies widely, the estimation should depend on parameters related to the distribution as well. Thus, which heterogeneity parameters of the distribution should be used and how can the parameters used help evaluate the mass–geometry interplay are questions of great interest that need to be answered by means of a precise mathematical treatment.

In Martín and Reyes [18] some asymptotic properties of the mass $\mu(A(\varepsilon))$ including the fractal dimension of the interface and the fractal dimension of the set where the mass of a surrounding multifractal distribution is concentrated, were obtained. However such set is not, in general, a closed set becoming somewhat intangible for the practical estimation of the measure. As a result, the effective estimation will require an adequate computer estimation method. This paper aims to present an interdisciplinary approach for the study of these kinds of problems.

The paper is organized as follows.

First, preliminary theoretical concepts an tools are presented in an accessible way for possible readers interested in the problem though not familiar with Fractal Geometry.

Second, the computation of the mass in different neighborhoods $\mu(A(\varepsilon))$ of the interface is implemented and its scaling behavior studied. This is first done for images of real river networks and later for proper fractal plane sets. The analysis of results allows one to study the influence of both, the irregularity of the geometric structure and the heterogeneity of the distribution, in the scaling of $\mu(A(\varepsilon))$.

Finally several applications for the analysis and modeling of fractal river networks structures and interplay between phases in granular and porous media are suggested.

2. Preliminary concepts

2.1. Fractal sets and dimensions

Mathematically speaking irregular fractal structures appear as sets of the Euclidean n-space $\mathbb{R}^n$ with intermediate size between smooth curves and surfaces, that is, sets of non integer Hausdorff dimension.

For $0 \leq s \leq n$, the s-dimensional Hausdorff measure of a set $A \subset \mathbb{R}^n$ is

$$H^s(A) = \liminf_{\delta \downarrow 0} \left\{ \sum_{i=1}^{\infty} d(S_i)^s \; : \; E \subset \bigcup_{i=1}^{\infty} S_i, \; d(S_i) \leq \delta \right\}.$$

In particular, the Hausdorff measure H^n is a constant multiple of the Lebesgue measure $\mathcal{L}^n$ (length, area, ...).

The Hausdorff dimension of a set $A \subset \mathbb{R}^n$ is defined by

$$\dim_H A = \inf \left\{ s \; : \; H^s(A) = 0 \right\} = \sup \left\{ s \; : \; H^s(A) = \infty \right\}$$

and indicates the level at which the measure of the set should be evaluated.

The somehow abstract concept of Hausdorff measure may be replaced by the Minkowski content which is better understood and numerically estimated.

Recall that for $0 < \varepsilon < \infty$ the closed ε-neighborhood of A is

$$A(\varepsilon) = \left\{ x \in \mathbb{R}^n : \; d(x, \; A) \leq \varepsilon \right\}.$$