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Existence and approximation for vibro-impact problems with a time-dependent set of constraints

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Abstract

We consider a discrete mechanical system subjected to perfect time-dependent unilateral constraints, which dynamics is described by a second order measure differential inclusion. The transmission of the velocity at impacts is given by a minimization property of the kinetic energy with respect to the set of kinematically admissible post-impact velocities. We construct a sequence of feasible approximate positions by using a time-stepping algorithm inspired by a kind of Euler discretization of the differential inclusion. We prove the convergence of the approximate trajectories to a solution of the Cauchy problem and we obtain as a by-product a global existence result.

Keywords: Discrete mechanical system; Time-dependent constraints; Differential inclusion; Inelastic shocks; Time-stepping scheme

1. Introduction

We consider a mechanical system with a finite number of degrees of freedom subjected to frictionless unilateral constraints. If we denote by $u(t)$ its representative point in generalized coordinates and by $K(t)$ the set of admissible positions at any instant t, the dynamics of the system is described by a second order differential equation combined with the condition $u(t) \in K(t)$, leading to a measure differential inclusion.

Such problems occur frequently in automotive industry or aeronautics where looseness of joints may create unwanted vibrations and impacts, but they are also of crucial importance in environment for the study of the damages due to earthquakes for instance. One can find an important literature on this topic in the case of time-independent constraints (i.e. when the set of admissible positions does not depend on t), which is more related to industrial issues, and several existence results have been obtained either by constructing a sequence of approximate trajectories and by proving their convergence to a solution of the Cauchy problem [24,6,4,5,23,11,20,9,14,22,15,10,7,8,17,19], or by using theoretical arguments based on existence results for ordinary differential equations and variational inequalities [1]. On the contrary, very few studies are available in the case of time-dependent constraints, which is more related to environmental issues like earthquakes. In [25] an existence result is established, by considering a generalization of the Yosida-type approximation already proposed in [20]. Unfortunately this technique transforms the differential

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inclusion into a sequence of very stiff ordinary differential equations which are not well suited for implementation (see [21] for a more detailed discussion). More recently another existence result, based on a time-stepping approximation of the problem, has been obtained [2] when the sets $K(t)$ are defined as a finite intersection of complements of convex sets. But this property is a very restrictive assumption and is usually not satisfied.

The aim of this work is thus to propose another time-stepping approximation of the problem and to state its convergence, which will give as a by-product a global existence result, in a more general geometrical setting than in [2]. More precisely we will first derive from the basic mechanical and geometrical description of the problem its mathematical formulation. Then we introduce a time-stepping scheme inspired by some implicit Euler's type discretization and the main steps of the convergence proof are outlined.

2. Formulation of the problem

We consider a discrete mechanical system which unconstrained motion is described by the following second order differential equation in $\mathbb{R}^d$:

$$\ddot{u} = g(t, u). \tag{1}$$

We assume that the system is subjected to time-dependent unilateral constraints i.e.

$$u(t) \in K(t) \quad \forall t \tag{2}$$

where $K(t)$ is the set of admissible positions at instant t, given by the geometrical inequalities

$$u \in K(t) \Longleftrightarrow f_\alpha(t, u) \geq 0, \quad \forall \alpha \in \{1, \dots, \nu\},\ \nu \geq 1$$

with smooth functions f_α. When a contact occurs, i.e. when $u(t) \in \partial K(t)$, a reaction force appears. If we assume that the contact is frictionless then [12]

$$\ddot{u} - g(t, u) = R \in -N\big(K(t), u\big) \tag{3}$$

where $N\big(K(t), u\big)$ is the normal cone to $K(t)$ at u given by

$$N\big(K(t), u\big) = \begin{cases} \{0\} \text{ if } u \in \mathrm{Int}\big(K(t)\big) \\ \Big\{ \sum_{\alpha \in J(t,u)} \lambda_\alpha \nabla_u f_\alpha(t, u),\ \lambda_\alpha \leq 0 \Big\} \text{ if } u \in \partial K(t) \\ \emptyset \text{ otherwise} \end{cases}$$

and $J(t, u) = \big\{\alpha \in \{1, \dots, \nu\};\ f_\alpha(t, u) \leq 0\big\}$ is the set of active constraints at the point (t, u).

As usual for vibro-impact problems, the velocity $\dot{u}$ may be discontinuous and the adequate framework for the solutions is the set of absolutely continuous functions u which derivative $\dot{u}$ belongs to the space of functions of bounded variation. Indeed, if we assume that the right and left velocities exist at some instant t, we infer from (2) that

$$\dot{u}^+(t) \in T\big(t, u(t)\big), \qquad \dot{u}^-(t) \in -T\big(t, u(t)\big)$$

with

$$T(t, u) = \big\{v \in \mathbb{R}^d;\ \partial_t f_\alpha(t, u) + \big\langle \nabla_u f_\alpha(t, u), v\big\rangle \geq 0 \text{ for all } \alpha \in J(t, u)\big\}.$$

It follows that $\dot{u}$ may be discontinuous at t if $J\big(t, u(t)\big) \neq \emptyset$ and the reaction force R is a measure. Of course the acceleration $\ddot{u}$ should now be understood as the Stieltjes measure $d\dot{u}$ and the measure $d\dot{u} - g(t, u)dt$ should vanish on $\big\{t;\ J\big(t, u(t)\big) = \emptyset\big\}$, in order to recover (1) when the constraints are not saturated. More precisely, there exist ν scalar measures λ_α such that

$$\begin{cases} d\dot{u} - g(\cdot, u)dt = \sum_{\alpha=1}^{\nu} \lambda_\alpha \nabla_u f_\alpha(\cdot, u) \\ \lambda_\alpha \geq 0, \quad \mathrm{Supp}(\lambda_\alpha) \subset \big\{t;\ f_\alpha\big(t, u(t)\big) = 0\big\} \quad \forall \alpha \in \{1, \dots, \nu\}. \end{cases}$$