Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 118 (2015) 329–342
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

Construction of univariate spline quasi-interpolants with symmetric functions[☆]

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Received 19 March 2014; received in revised form 20 August 2014; accepted 20 November 2014
Available online 5 January 2015

Abstract

A new approach to construct univariate spline quasi-interpolants on arbitrary partitions of bounded intervals is developed. In the first part of this paper, we give some results about the symmetric functions of the difference of two finite sets. These results are used, in the second part of this work, to construct explicitly different types of quasi-interpolants. We revise the definition of a uniformly bounded quasi-interpolant and we propose some results on this subject. Some numerical examples are given to illustrate our theoretical results.

Keywords: Symmetric functions; Quasi-interpolants; Uniformly bounded operators

1. Introduction

Let $[a, b]$ be a bounded interval in $\mathbb{R}$, n a non negative integer and let

$$\tau = \{a = x_0 < x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_{n-1} < x_n = b\}$$

be an arbitrary subdivision of $[a, b]$. In order to define B-splines of degree k, where k is a non negative integer, we add at the extremities of the interval $[a, b]$ the knots $x_{-k} = x_{-k+1} = \cdots = x_0 = a$ and $x_{n+k} = x_{n+k-1} = \cdots = x_n = b$. The knot set obtained by adding these knots to τ is denoted by τ_k. Moreover we suppose that

$$x_{i+k} - x_i > 0 \quad \text{for all } i = -k + 1, \ldots, n - 1.$$

We denote by $\mathcal{S}_k([a, b], \tau_k)$ the space of continuous spline functions of degree k and by $\{B_{i,k}, i = -k, \ldots, n - 1\}$ the B-spline basis of $\mathcal{S}_k([a, b], \tau_k)$. Notice that each B-spline $B_{i,k}$ is supported in the bounded interval $[x_i, x_{i+k+1}]$ (see [5], for instance).

It is well known that quasi-interpolation is a general approach to construct, with low computational cost, efficient local approximants to a given set of data or a given function. Quasi-interpolation has received a considerable attention by many authors (see [1,5,4,2,3,6–8,11,12,10,13] and references therein). The univariate spline quasi-interpolants

☆ Research supported by URAC-05.
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http://dx.doi.org/10.1016/j.matcom.2014.11.028
0378-4754/

(abbr.QIs) of degree k are defined as operators of the form:

$$Q_k(f) = \sum_{i=-k}^{n-1} \lambda_{i,k}(f) B_{i,k},$$

where $\lambda_{i,k}$ are linear functionals. When $\lambda_{i,k}(f)$ is a linear combination of discrete values of f at some points $y_{i,j}$, $1 \le j \le k+1$, lying in a neighborhood of $supp(B_{i,k})$, the corresponding QI is called discrete QI, and when $\lambda_{i,k}(f)$ is a linear combination of values of derivatives of f at some point y_i, lying in $supp(B_{i,k})$, the corresponding QI is called differential QI (see [5,4,12,10], for instance).

For a given positive integer $p \le k$, we recall that the operator Q_k is said to be exact on the space $\mathbb{P}_p$ of polynomials of degree at most p, provided that $Q_k(q) = q \ \forall q \in \mathbb{P}_p$. It follows that Q_k has an optimal approximation order whenever $p = k$. Throughout this paper we denote such operators by $Q_k^{(p)}$. The main advantage of quasi-interpolant operators is that they can have an optimal approximation order with a simple and direct construction without solving any system of linear equations. Moreover, they are local in the sense that $Q_k^{(p)}(f)(x) = \sum_{i=-k}^{n-1} \lambda_{i,k}^{(p)}(f) B_{i,k}(x)$ only depends on values of f in a neighborhood of x. Finally, notice that these approximants have many applications to numerical analysis as integration, differentiation, approximation of zeros and solutions of boundary problems.

The first aim of this work is developed in the second section. So we introduce some mathematical backgrounds which are useful for computing the coefficient functionals $\lambda_{i,k}^{(p)}(f)$. For this, we use the well known symmetric functions of a finite set A to establish new results concerning the symmetric functions of the difference of two finite sets A and B. These results have a direct application in the computation of the above coefficients.

The second aim of this paper is developed in Section 3. We give a general method to build univariate spline QIs of different types in the space $\mathcal{S}_k([a,b], \tau_k)$. Then, we establish the analog of the classical results on interpolation operators such as Lagrange form, Newton form and osculatory quasi-interpolation for the coefficients $\lambda_{i,k}^{(p)}(f)$. On the other hand, we give the explicit formulae allowing the computation of these coefficients in their different forms. In Section 4, the definition of a uniformly bounded quasi-interpolant is revised. In this context, we propose some schemes on the data sites in order to obtain uniformly bounded QIs $Q_k^{(p)}(f)$, where the infinite norm of the quasi-interpolant operator $Q_k^{(p)}$ must be bounded independently of the data sites. We show that if the quasi-interpolation points $y_{i,j}$ are appropriately chosen near $supp(B_{i,k})$, then $Q_k^{(p)}$ is uniformly bounded. In Section 5, some uniformly bounded quadratic and cubic discrete QIs are presented. To illustrate our theoretical results, we give in Section 6, some numerical examples.

2. Symmetric functions

Let n be a positive integer and let $A_n = \{a_1, a_2, \dots, a_n\}$ be a finite set of real numbers. Throughout this paper, we denote by ψ_{A_n} the following polynomial of degree n:

$$\psi_{A_n}(x) = \prod_{i=1}^{n}(x - a_i). \tag{2.1}$$

If we denote by $[A_n]$ the row matrix $[a_1, \dots, a_n]$, then we define

$$\psi_{[A_n]} = \psi_{A_n}. \tag{2.2}$$

Definition 1. Let l be a positive integer such that $0 \le l \le n$. Here, we define the symmetric function of order l of the set A_n by

$$\theta^{[l]}(A_n) = (-1)^l \frac{l!}{n!} \psi_{A_n}^{(n-l)}(0). \tag{2.3}$$

For given finite sets E and F such that $\sharp E \le \sharp F$, where $\sharp E$ is the cardinality of the set E, we denote by $\mathcal{I}(E, F)$ the set of all injections from E into F. Note that $\sharp\mathcal{I}(E, F) = \frac{f!}{(f-e)!}$ where $e = \sharp E$ and $f = \sharp F$.

For all integers $m, k \ge 1$, we set

$$I_k = \{1, 2, \dots, k\} \quad \text{and} \quad B_m = \{b_1, \dots, b_m\}.$$