Available online at www.sciencedirect.com
SciVerse ScienceDirect
Mathematics and Computers in Simulation 96 (2014) 4–53
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original article

Observations on evolutionary models with (or without) time lag, and on problematical paradigms

Christopher T.H. Baker [a,b,*]

[a] *Dept. of Mathematics, University of Chester, CH1 4BJ, UK*
[b] *School of Mathematics, University of Manchester M13 9PL, UK*

Received 2 July 2011; received in revised form 10 June 2012; accepted 14 July 2012
Available online 27 August 2012

In memoriam Prof. F.J. Ursell, FRS (1923–2012).

Abstract

Models of phenomena that evolve with time, notably those in bioscience, may be more realistic as scientific models if they incorporate a memory or time-lag rather than assume instantaneous effects. The differences and similarities between the two types of model, having regard to their 'faithfulness' as models, the sensitivity, stability, and qualitative behaviour of their solutions, are often discussed through canonical models or suggested paradigms using various 'mathematical toolkits'. We offer insight (largely self-contained, but supplemented by a bibliography) into mathematical models with time-lag and we consider how appropriate or limited are various exemplars when used as paradigms. We also mention possibilities for extending the application of the mathematical tools at our disposal.

Keywords: Canonical forms; Mathematical models; Paradigms; Qualitative behaviour; Sensitivity

'Explicit expressions of solutions are unavailable for the most interesting differential equations!' M. Rasmussen [136]

'These things are well-known to those who know them!' F.J. Ursell [informal comment]

1. Prologue

Purpose: In this prologue we seek to provide an introduction to, and a perspective on, the remainder of the discourse, and we indicate the intended (fairly broad) readership. Every subsequent section has a similar introduction to its content.

1.1. Introduction

The problems of determining vector-valued functions y of the real variable t such that

$$\mathrm{y}'(t) = \mathrm{f}_{\natural}(t, \mathrm{y}(t)) \quad (t \geq t_0) \quad \text{subject to} \quad \mathrm{y}(t_0) = \mathrm{y}_0, \tag{1.1}$$

* Correspondence address: Dept. of Mathematics, University of Chester, CH1 4BJ, UK.
E-mail addresses: Christopher.Baker@manchester.ac.uk, c.baker@chester.ac.uk, cthbaker@na-net.ornl.gov, cthbaker@ma.man.ac.uk

0378-4754/$36.00 © 2012 IMACS. Published by Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.matcom.2012.07.017

or such that

$$\mathrm{y}'(t) = \mathrm{f}(t, \mathrm{y}(t), \mathrm{y}(t-\tau)) \quad (t \geq t_0) \quad \text{subject to} \quad \mathrm{y}(t) = \phi(t) \quad \text{for } t \in [t_0 - \tau, t_0], \tag{1.2}$$

are relatively simple illustrations of the types of evolutionary problem that we consider.

Remark i. Sequentially numbered *Remarks* incorporated in the text have the same rôle as extended footnotes. Lower case symbols in 'Roman' or 'upright Greek' fonts always denote vectors or vector-valued functions; corresponding upper case fonts denote matrices. $\mathbb{R}$ denotes the field of real numbers; $\mathbb{C}$ denotes the complex numbers, and we use $\mathbb{E}$ to denote either $\mathbb{R}$ or $\mathbb{C}$ when there is a choice between them – this choice to be made consistently, in a given statement. To avoid an abuse of notation we make the natural identification so that $\mathbb{R}^n \subset \mathbb{C}^n$. $\mathbb{E}^n$ is Euclidean space, with $\mathbb{E} = \mathbb{R}$ or $\mathbb{E} = \mathbb{C}$, and we refer to n as the (spatial) dimension, though n may exceed 3 and need not refer to physical space. See below, for further notation.

We impose suitable assumptions on $\mathrm{f}_\natural$, f, ϕ and assume $t_0 \in \mathbb{R}$, $\mathrm{y}_0 \in \mathbb{E}^n$, $\tau > 0$, $\phi \in ([t_0 - \tau, t_0] \to \mathbb{E}^n)$, $\mathrm{f}_\natural \in ([t_0, \infty) \times \mathbb{E}^n \to \mathbb{E}^n)$ and $\mathrm{f} \in ([t_0, \infty) \times \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{E}^n)$, $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ (where $(X \to Y)$ denotes the functions mapping a set X to a set Y). For (1.1) and (1.2), we seek $\mathrm{y} \in ([t_0, \infty) \to \mathbb{E}^n)$. Where (1.1) or (1.2) represents a real-life model, t often represents time, and the initial (natural) choice of $\mathbb{E}$ is $\mathbb{R}$. Problem (1.1) involves finding a solution of an ordinary differential equation (ODE) that assumes a given initial value while (1.2) involves finding the solution of a retarded- or delay-differential equation (RDE or DDE) that agrees with an 'initial function', defined on an 'initial domain': for (1.2) the initial domain is $[t_0 - \tau, t_0]$. In (1.1) $\mathrm{y}'(t)$ depends on $\mathrm{y}(t)$ while in (1.2) it depends also on the "previous" value $\mathrm{y}(t-\tau)$; (1.1) is an *initial-value problem* while (1.2) is an *initial-function problem*. The equations

$$\mathrm{y}'(t) = \mathrm{f}_\natural(\mathrm{y}(t)) \quad \text{and} \quad \mathrm{y}'(t) = \mathrm{f}(\mathrm{y}(t), \mathrm{y}(t-\tau)) \tag{1.3}$$

are the 'autonomous' (time-invariant) versions of the differential equations in (1.1) and (1.2).

We are motivated by models in bioscience. We seek to discover or rediscover, interpret or organize, and apply – in a multidisciplinary spirit – theoretical results related to modelling and computational modelling. In mathematical modelling (regarded as a subset of applied mathematics), 'size' and 'scale' can be more significant than in pure mathematics, but there is scope for applied and pure mathematicians to learn from each other. Deterministic models of evolutionary phenomena often incorporate ODEs, DDEs, integral equations, integro-differential equations – or their counterparts using differences and sums or using partial derivatives; for a brief overview see the introduction in [15]. Our interest is in the types of ODEs and DDEs (and similar equations – see Section 2.1) that are used as models in biomathematics or are used as exemplars in analysis, numerical analysis, or computation.

Notwithstanding the length of our text, we have had to make some omissions. Thus, we refer only briefly to partial differential equations (PDEs) and omit neutral delay differential equations (NDDEs; cf., e.g., [80]) from our discussion. Many of the deterministic equations referred to have stochastic counterparts (cf., e.g., [151]) but we study only deterministic problems.

Remark ii. For some time, applied mathematics papers that related to the solution of DDEs opened with an indication of their areas of applicability in science, medicine and engineering. Given their history over a number of decades, the existence of many texts detailing their properties, their use as models and their numerical solution (and the availability of mathematical software for their numerical solution), we feel no need for an extended justification of their study. There exist many examples of the successful use in medicine and biomathematics of models involving differential equations with time lags and we might hope that the mathematically aware reader has already encountered such problems. To quote Engelborghs et al. [59],

> "Dynamical systems with time delays have been studied for more than two centuries, dating back to Euler, but most progress has occurred in the twentieth century, with the significant contributions of Lotka and Volterra. Although there is now a substantial body of theory available, the global knowledge of delay equations has not been widely exploited by the scientific community. This is somewhat changing, nowadays, with a rapidly growing use of systems with delays in applied sciences, most notably mathematical biology and engineering. The main advantage of explicitly incorporating time delays in modeling equations is to recognize the reality of non-instantaneous interactions."

The observation that, particularly in biomathematics, models incorporating a memory or time-lag may be more realistic than models based on instantaneous effects, prompts an assessment of some of the similarities and differences