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Original article

Application of symmetry and singularity analyses to mathematical models of biological systems

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Abstract

Common mathematical tools for the analysis of the differential equations which arise in models of biological systems are the theory of dynamical systems and numerical analysis. We provide a number of examples which demonstrate the utility of including symmetry and singularity analyses as part of one's standard repertoire when investigating such models.

Keywords: Dynamical systems; Lie symmetries; Singularity analysis; Computational methods

1. The four elements of analysis

- **Dynamical systems**: One of the two traditional approaches and yields valuable information about the behaviour of systems of equations whether they be integrable in closed form or not.
- **Computational methods**: The second traditional approach. Can enter the investigation at any stage. Can be very expensive if employed willy-nilly.
- **Lie symmetries**: Developed by Sophus Lie around 1870 [40–45]. Based upon the concept of invariance of, say, a system of differential equations under transformation. Uses infinitesimal transformations which means that the differential equations to be solved are linear.
- **Singularity analysis**: Initiated by Kowalevskaya in 1888 [33] and greatly developed since then [55,20–22,3–6,9,12–17,28]. Its essence is the expansion of the solution about a polelike singularity in terms of a Laurent series.

These approaches have different emphases and directions of investigation, but they should not be regarded as completely independent.

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2. Dominant behaviour of quadratic systems

2.1. Preamble

Quadratic systems have the form

$$\dot{x}_i = a_{ij}x_j + b_{ijk}x_jx_k, \quad i, j, k = 1, n. \tag{1}$$

Typically a_{ij} is diagonal and represents the most elementary model of population growth which was introduced by Malthus in 1798 [47]. The quadratic terms include the term introduced by Verhulst in 1838 [60], known as the logistic term. The interaction with other 'species' was introduced in the context of chemical reactions by Lotka in 1925 [46] and in the context of the populations of competing species of fishes by Volterra in 1926 [61]. This type of representation of the interaction of two components of a system has been extended to many areas in Ecology, Epidemiology, Economics and Medicine. As Murray observes [53] [p. 327] in his discussion of a model of the incidence of gonorrhoea, an interaction term of this type implicitly assumes free mixing of the elements of each component of the system. The possibility of explicit time-dependence in the parameters is frequently admitted in the statement of the model to be investigated and then abandoned in the analysis since the time-dependence plays havoc with the standard methods of dynamical systems (see, for example, Murray's analysis of a model for HIV-AIDS [53]). Lotka–Volterra and the related quadratic systems have received much attention, mainly from the viewpoint of dynamical systems for which they are textbook examples (see, for example, Murray [53] [Chap. 3]). Other investigations have been based on the Carleman method [7,8] and a combination of Lie and Painlevé analyses [32]. Golubchik and Sokolov [24,25] introduced a class of integrable quadratic systems in the context of the Yang–Baxter equations and Cotsakis et al. [34] analysed exemplars of this class in terms of their singularity and symmetry analyses.

We examine a class of equations of the type of (1) in terms of the properties of the dominant terms in the system and consider the specific system (see also [54])

$$\dot{x} = x(x + by) \tag{2}$$

$$\dot{y} = y(ax + y) \tag{3}$$

in which a and b are constant parameters. The linear terms of (1) are not dominant terms. We allow for self-interaction terms and cross-species-interaction terms as factors affecting the rate of change of the two dependent variables. We do not consider the possibility of other-species-interaction terms as affecting the rate of change. The number of possible parameters in the system is reduced to two by means of rescalings of the variables. The derivatives and quadratic terms of (1) are the dominant terms of the system in terms of singularity analysis. The system ((2) and (3)) is automatically integrable in the sense of Lie since it possesses the two obvious point symmetries of invariance under time translation and rescaling. Our aim is to investigate the conditions under which the quadratic system is integrable in the sense of Painlevé for it is from the integrability of the basic quadratic system that the possible integrability of the full system follows. We examine the system ((2) and (3)) in terms of its singular behaviour both as the system stands and as it can be rewritten in terms of a single second-order equation. We identify specific values of the parameters, a and b, for which the Painlevé criteria for integrability are satisfied. The system, as written in terms of a single second-order equation, is examined for its Lie point symmetries and again critical values of the parameters are identified.

3. Singularity analysis of the two-dimensional system

We commence our investigation with the singularity analysis of the given system, ((2) and (3)). To obtain the leading-order behaviour we substitute

$$x = \alpha\tau^p \quad y = \beta\tau^q \tag{4}$$

into ((2) and (3)) to obtain

$$\alpha p\tau^{p-1} = \alpha^2\tau^{2p} + b\alpha\beta\tau^{p+q}$$

$$\beta q\tau^{q-1} = a\alpha\beta\tau^{p+q} + \beta^2\tau^{2q}$$