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Computational analysis of the conserved curvature driven flow for open curves in the plane

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Abstract

The paper studies the constrained curvature flow for open planar curves with fixed endpoints by means of its numerical solution. This law originates in the theory of phase transitions for crystalline materials and where it describes the evolution of closed embedded curves with constant enclosed area. We show that the area is preserved for open curves with fixed endpoints as well. Here, the area is given by the curve and its ends connected to the origin of coordinates. We provide the form of the stationary solution towards which any other solution converges asymptotically in time. The evolution law is reformulated by means of the direct method into the system of degenerate parabolic partial differential equations for the curve parametrization. This system is spatially discretized by means of the flowing finite volumes method and solved numerically by the explicit Runge–Kutta solver. We experimentally investigate the order of approximation of the scheme by means of our numerical data and by knowing the analytical solution. We also discuss the role of the suitable tangential redistribution. For this purpose, several computational studies related to the open curve dynamics are presented.

Keywords: Phase transitions; Area-preserving curvature flow; Parametric method; Open curves; Experimental order of convergence

1. Introduction

In this article we focus on the non-local curvature flow for open curves in $\mathbb{R}^2$. Our main goal is to investigate the flow described by the following geometric evolution law:

$$v_\Gamma = -\kappa_\Gamma + F, \quad \text{where } F = \frac{1}{L(\Gamma_t)} \int_{\Gamma_t} \kappa_\Gamma \mathrm{d}s, \tag{1}$$

$$\Gamma_t|_{t=0} = \Gamma_{ini}, \tag{2}$$

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where Γ_t is a C^1 smooth open curve with fixed endpoints in $\mathbb{R}^2$. It is evolved in the normal direction with the velocity v_Γ. The evolution starts from the initial curve Γ_{ini}. Here $L(\Gamma_t) = \int_{\Gamma_t} \mathrm{d}s$ is the length of the curve Γ_t and κ_Γ is the (mean) curvature of Γ_t.

In the case where Γ_t is a closed Jordan curve, $\mathbf{n}_\Gamma$ is the outward unit normal vector and the curve is assumed to be oriented counter-clockwise. This means, that $\kappa_\Gamma = 1$ if Γ_t is the unit circle. In the other cases, i.e., the case where Γ_t is an open curve or a self-intersected closed curve, $\mathbf{n}_\Gamma$ is chosen in such a way that $\det(\mathbf{n}_\Gamma, \mathbf{t}_\Gamma) = 1$ where $\mathbf{t}_\Gamma$ is the unit tangent vector to the Γ_t (see Section 2).

Geometric laws similar to (1) have been discussed in the literature (see [14,11,20,13]) or [18]. They belong to a class of (mean) curvature flows described by the evolution law

$$v_\Gamma = -\kappa_\Gamma + F, \tag{3}$$

with a particular choice of the forcing term F, which is widely studied in the literature (see, e.g., [12]) The evolution of open curves has been addressed, e.g., in [5,22] or in [23].

The global character of the forcing term F often plays its role in the constrained motion by (mean) curvature, where the F depends on geometrical properties of Γ_t, like its length $L(\Gamma_t)$, enclosed area $A(\Gamma_t)$ etc. The particular choice of F as in (1), i.e., $F = \int_{\Gamma_t} \kappa_\Gamma \mathrm{d}s / L(\Gamma_t)$, leads to the area preserving (mean) curvature flow, whereas $F = \int_{\Gamma_t} \kappa_\Gamma^2 \mathrm{d}s / \int_{\Gamma_t} \kappa_\Gamma \mathrm{d}s$ yields the length preserving (mean) curvature flow (see [30]), or, by choosing the force as $F = L(\Gamma_t)/2A(\Gamma_t)$, we can investigate the isoperimetric ratio gradient flow (see [30]).

The local character of F is often observed in applications of the (mean) curvature flow in digital image processing. Namely in image segmentation, where the force F locally depends on the intensity of the segmented image (see, e.g., [3,4]).

The (mean) curvature flow with a particular choice of the forcing term F found its applications in many problems with physical context, e.g., in dislocation dynamics in crystalline materials, where F can describe either global stress field or local interaction forces between multiple defects (see [6,22]). The constrained motion by (mean) curvature, in particular, has been investigated in [27,17,7] within the context of a modification of the Allen–Cahn equation (see [8,1]) approximating the (mean) curvature flow (see [2]). The non-local character of the geometric governing equation is also connected to the recrystallization phenomena where a fixed previously melted volume of the liquid phase solidifies again (see [19]).

Problem (3) for closed curves can be mathematically treated by the direct (parametric) method (see, e.g., [10,12,4]), by the level set method (see, e.g., [24]) or by the phase-field method (see, e.g., [2]). In this paper, we investigate (3) by means of the direct method as the single option for open or self-intersecting curves and solve the resulting degenerate parabolic system numerically to provide the information on the solution behavior. For this purpose, the used numerical scheme based on the flowing finite volume method is suggested using the previous authors' experience. Approximation property of this scheme is analyzed and the role of the redistribution for its stable behavior is explained. Then, several computational examples are presented.

2. Equations

In the direct method for solving (1) one considers the parametrization of the smooth time-dependent curve Γ_t for $t \geq 0$ by means of the mapping

$$\vec{X} = \vec{X}(t, u), \quad u \in [0, 1],$$

where u is the parameter in a fixed interval. In the case of a closed curve, the parametrization is orientated counter-clockwise and the periodic boundary conditions at $u = 0$ and $u = 1$ are imposed, i.e. $\vec{X}(t, 0) = \vec{X}(t, 1)$. For open curves with fixed ends we prescribe the Dirichlet boundary conditions for $\vec{X}(t, u)$ at $u = 0$ and $u = 1$; i.e. $\vec{X}(t, 0) = \vec{X}_0$ and $\vec{X}(t, 1) = \vec{X}_1$ for given positions $\vec{X}_0$ and $\vec{X}_1$, respectively. Consequently, the geometric quantities of our interest can be expressed by means of the mapping $\vec{X}$. The unit tangent and normal vectors are given by the following formulas:

$$\mathbf{t}_\Gamma = \frac{\partial_u \vec{X}}{|\partial_u \vec{X}|} \quad \text{and} \quad \mathbf{n}_\Gamma = \frac{\partial_u \vec{X}^\perp}{|\partial_u \vec{X}|} = \frac{1}{|\partial_u \vec{X}|} \begin{pmatrix} \partial_u X^2 \\ -\partial_u X^1 \end{pmatrix}, \quad \text{where } \vec{X} = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$$