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Two-dimensional hp adaptive finite element spaces for mixed formulations

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Abstract

One important characteristic of mixed finite element methods is their ability to provide accurate and locally conservative fluxes, an advantage over standard H^1-finite element discretizations. However, the development of p or hp adaptive strategies for mixed formulations presents a challenge in terms of code complexity for the construction of $\mathbf{H}(div)$-conforming shape functions of high order on non-conforming meshes, and compatibility verification of the approximation spaces for primal and dual variables (inf–sup condition). In the present paper, a methodology is presented for the assembly of such approximation spaces based on quadrilateral and triangular meshes. In order to validate the computational implementations, and to show their consistent applications to mixed formulations, elliptic model problems are simulated to show optimal convergence rates for h and p refinements, using uniform and non-uniform (non-conformal) settings for a problem with smooth solution, and using adaptive hp-meshes for the approximation of a solution with strong gradients. Results for similar simulations using H^1-conforming formulation are also presented, and both methods are compared in terms of accuracy and required number of degrees of freedom using static condensation.

Keywords: $\mathbf{H}(div)$-conforming finite elements; Mixed method; hp-Adaptivity

1. Introduction

Mixed finite element formulations are usually employed for fluid flow simulations in heterogeneous media [3]. They are based on simultaneous approximations of the primal (pressure) and dual (flux) variables, involving two kinds of approximation spaces. Approximation subspaces in

$$\mathbf{H}(div, \Omega) = \left\{ \mathbf{q} \in \left[L^2(\Omega)\right]^d ; \nabla.\mathbf{q} \in L^2(\Omega) \right\}$$

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are required for the flux variable. In order to be a function in $\mathbf{H}(div, \Omega)$, the pieces $\mathbf{q}^K = \mathbf{q}|_K$ over elements K of a partition of Ω should be assembled by keeping continuous normal components across common element edges. Discontinuous finite element spaces are used for pressure approximations.

The development of this method presents a challenge in terms of compatibility verification of the approximation spaces for primal and dual variables (*inf–sup* condition), and of code complexity for the construction of high order $\mathbf{H}(div)$-conforming basis functions. Since the pioneering work by Raviart and Thomas [14] in 1977, different constructions of $\mathbf{H}(div)$ (or $\mathbf{H}(curl)$) approximation space have been proposed in [2,12]. In some contexts the vector basis functions are constructed on the master element and then they are transformed to the elements of the partition by Piola transformations, as described in [3,15,1]. Recent constructions of hierarchical high order spaces, as described in [18,19,5], are based on the properties of the De Rham complex.

A different methodology for the construction of vectorial shape functions for $\mathbf{H}(div)$-conforming approximation spaces is proposed in [16]. They are based on conformal rectangular and triangular affine partitions (without hanging edges) and uniform polynomial degree distribution of arbitrary order. The principle is to choose appropriate constant vector fields, based on the geometry of each element, which are multiplied by an available set of H^1 hierarchical scalar basic functions. In the present paper, this methodology is extended to non-conformal meshes and varying degree distributions. As in the case of conformal meshes, the assembly of the resulting vector basis functions to obtain continuous normal components on the element interfaces is a direct consequence of the properties of the properly chosen vector fields and of the continuity of the scalar basic functions, which is described in [4] for *hp*-adapted meshes, without limitations on hanging sides and distribution of approximation orders.

All the implementations of the present paper are performed in the object-oriented scientific computational environment called NeoPZ,[1] developed at the Universidade Estadual de Campinas (Brazil) by the research group under the leadership of P.R.B. Devloo, at the Laboratory of Computational Mechanics. NeoPZ is a general finite element approximation software organized by modules for a broad class of technologies, incorporating a variety of element geometries, variational formulations, and approximation spaces (e.g. continuous [9], discontinuous [10], $\mathbf{H}(div)$-conforming [16], and others). It allows the user to apply *hp*-strategies by choosing locally the mesh refinement and the order of approximation [4]. NeoPZ is integrated with pthreads and thread building blocks for efficient execution on multi core computers. Multiphysics simulations can also be implemented by combining different approximation spaces into a coupled system of equations [11], a procedure that facilitates the implementations of mixed formulations based in different approximation spaces for dual and primal variables. In fact, the current proposed methodology for the construction of $\mathbf{H}(div)$-conforming functions is designed having in mind the resources provided by NeoPZ, where the required hierarchical high order continuous scalar basis functions are already implemented for conformal or non-conformal *hp*-meshes.

The implementation of *hp*-adaptive strategies is more complex than for standard finite element schemes [7,8,18,17]. The differences in polynomial orders and the presence of hanging sides introduce additional difficulties in the enforcement of the constraints of continuous normal components. This is a reason for the fact that there exist few computational environments available to the scientific finite element community which implement *hp*-adaptive approximation spaces, specially for mixed formulations. The group under the leadership of L. Demkowicz, at the University of Texas at Austin, has developed the *hp90* package [6] supporting simulations in *hp* spaces of H^1, $\mathbf{H}(div)$ and $\mathbf{H}(curl)$ type. See also [13], for application of mixed *hp* finite method to linear elasticity problems. The group working with P. Šolín, at the University of Nevada at Reno, has developed the system called HERMES[2] implementing $\mathbf{H}(curl)$ calculations.

The paper is organized as follows. Section 2 describes the assembling process for the construction of $\mathbf{H}(div)$-conforming subspaces in non-conformal mesh (i.e., with hanging nodes) with arbitrary degree distribution, by first summarizing the main steps for the construction of vectorial shape functions for them. The main properties of the required hierarchical scalar H^1-conforming bases, and the family of constant vector fields can be found in [16], but they are summarized in the Appendix. In order to validate the assembly of the *hp*-approximation $\mathbf{H}(div)$-conforming spaces, and to show their consistent applications to mixed formulations in quadrilateral and triangular meshes, two model problems are discussed in Section 3. The first one has smooth solution, which is used to show optimal convergence rates for *h*-refinement simulations using uniform and non-uniform (non-conformal) settings. The other one has

[1] http://github.com/labmec/neopz.
[2] http://hpfem.org/hermes.