Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 117 (2015) 20–38
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

Asymptotic dynamics of a piecewise smooth map modelling a competitive market

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Received 7 January 2015; received in revised form 13 May 2015; accepted 22 May 2015
Available online 16 June 2015

Abstract

In the present work we study asymptotic dynamics of a multi-dimensional piecewise smooth map which models an oligopoly market where competitors use adaptive scheme for reaction choice. Each competitor also defines the moment for renewing the capital equipment depending on how intensively the latter is used. Namely, the larger output is produced, the quicker the capital exhausts. It is shown then that the asymptotic dynamics of the map allows coexistence of different metric attractors in which case it is sensitive to initial conditions. We also investigate stability of trajectories representing Cournot equilibria which are here not fixed but periodic points. In particular, it is shown that several such Cournot equilibria, belonging to different invariant manifolds, may coexist some of them being locally asymptotically stable and some being unstable.

Keywords: Multidimensional piecewise smooth map; Coexisting metric attractors; Oligopoly market model; Cournot equilibrium stability

1. Introduction

Main purpose of this paper is to study asymptotic dynamics of the multi-dimensional iterative system modelling an oligopoly market that is supposed to develop into perfect competition economy with increasing the number of players involved. For having such a transition two conditions must hold: First, the Cournot equilibrium must seamlessly turn into the competitive equilibrium. Second, the Cournot equilibrium must not be destabilised through the addition of new competitors. The first condition does generally not present any problem, but the latter has been seriously questioned, which is also known as the Theocharis problem [25]. However, it should be noted that this "Theocharis problem" was fully analysed 20 years earlier by Palander [13]. The mentioned stability issue is the following. Assume we deal with a linear demand function and Cournot oligopolists facing constant marginal costs. Then Cournot equilibrium becomes locally unstable when the number of competitors crosses a rather low threshold of three. Moreover, if one wants to consider the global dynamics (which neither Palander, nor Theocharis did), it is necessary to take into account that

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http://dx.doi.org/10.1016/j.matcom.2015.05.004

such a linear model being unstable is bound to result in negative outputs, so, to make sense, it must be constrained. If so, the model becomes piecewise linear which was considered, e.g., in [6,19].

In [16] (developed also later in [17]) one of the present authors suggested an alternative demand function, an isoelastic one, where price and demand were just reciprocal. This isoelastic demand function has a certain attraction because it automatically results whenever the utility function is of the Cobb–Douglas type. However, as shown by [1,2], the same (Theocharis) problem as with a linear demand function occurs as well with an isoelastic demand function whenever we assume constant marginal costs. The only difference is that the watershed between stability and instability is raised from three to four firms.

The problem seems to arise not from the demand function but from constant marginal costs (constant returns), which means assuming firms that are potentially infinitely large in capacity. It is then neither surprising nor interesting that the addition of competitors has a destabilising effect. Further, it is not the comparison we want to make. We want to compare the case with *many small firms* to the case with a *few large firms*. This is impossible to represent unless we assume capacity limits as suggested already by Edgeworth [7].

The most convenient way to introduce capacity limits using standard economic theory is to use the constant elasticity of substitution (CES) production functions, and some ideas from "putty/clay" modelling in growth theory, though, unlike [9], only take capital as fixed. The fixed capital provides a natural capacity limit which smoothly drives variable costs to infinity when the limit is approached. Together with the assumed isoelastic demand function it results in a convenient case where we can solve for explicit reaction functions (see [22]). In this way one can specify the number of firms and any given total capacity for the industry to be split among the firms. One of the present authors tried his hands on this, [20], obtaining the result that the destabilisation might be avoided.

With introducing capacity limits like above one is restricted to considering reaction functions of two different forms—in the short run (with existing capacity limit) and in the long run (with constant returns to scale). Switching from the short run function to the long run one happens at the moment when the firm renews its capital (makes a re-investment), in the next iteration the short run branch is then restored. Moreover, these reinvestment moments may disagree for different competitors. This leads to a rather complicated multidimensional piecewise smooth dynamical system.

Multidimensional dynamical systems defined by smooth maps are already rather knotty to investigate analytically, in particular, due to that they can be both expanding and contracting on the same invariant set. This often gives rise to complicated long-term behaviour such as Smale horseshoe [23] or Smale–Williams solenoid [28]. Non-smooth systems are even more complex with respect to smooth ones since their state space consists of several partitions separated by borders (switching manifolds) at which the map is not differentiable. These are closely related to the border collision bifurcations which may cause an abrupt change of the orbit, for instance, direct transition from a stable fixed point to chaos (see, e.g., [4,12,24] and references therein). To study such complex systems mainly numerical simulations are used except some particular cases when the map can be reduced to simpler form. Apart from this certain aspects of asymptotic behaviour of the map can be described by means of various cumulative entities known from statistical methods and ergodic theory [27].

In the current paper we develop the ideas expressed already in [14,15]. In these previous articles the fixed lifetime of capital equipment was exogenous which resulted in a non-autonomous dynamical system. We now endogenise the investment decision and let production being under or above the optimal capacity lengthen or shorten the remaining lifetime of capital.

2. The model

Suppose there are n competing firms in the market each producing q_i amount of the same commodity (*individual supply*). Then $Q = \sum_{i=1}^{n} q_i$ denotes the *total supply* equal to the total market demand in Cournot equilibrium. We also introduce a *residual supply* for every firm as $Q_i = Q - q_i$, $i = \overline{1, n}$.

2.1. Demand

Assume the isoelastic inverse demand function (as in [16]):

$$p = \frac{1}{Q} = \frac{1}{Q_i + q_i}, \quad i = \overline{1, n}, \tag{1}$$