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Original Article

Equivalence conditions for the finite volume and finite element methods in spherical coordinates

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Abstract

A numerical technique for the solution of the compressible flow equations over unstructured grids in a spherical reference is presented. The proposed approach is based on a mixed finite volume/finite element discretization in space. Equivalence conditions relating the finite volume and the finite element metrics in spherical coordinates are derived. Numerical simulations of the explosion and implosion problems for inviscid compressible flows are carried out to evaluate the correctness of the numerical scheme and compare fairly well to one-dimensional simulations over very fine grids.

Keywords: Compressible flows; Shock waves; Explosion/implosion problem; Spherical coordinates; Finite element/volume methods

1. Introduction

In a spherical reference, diverse gasdynamics problems exhibit relevant symmetries. These are, e.g., detonations, astrophysical flows, Inertial Confinement Fusion (ICF) applications, sonoluminescence phenomena and nuclear explosions [13]. To compute the numerical solution of the compressible flow equations for these kind of flows, an interesting possibility is provided by the use of a mixed finite volume (FV)/finite element (FE) approach [6]. For example, in viscous flows, it is possible to use the FV and the FE to compute the advection and dissipation terms, respectively, within the same algorithm [12,1,3]. The typical approach to build such methods is to evaluate the fluxes of the Euler equations by a classical stabilized node-centred FV scheme and to exploit the FE viewpoint to discretize the viscous or diffusion terms of the Navier–Stokes equations as well as to possibly estimate the solution gradients, needed by high order reconstruction schemes [14]. Such a possibility is expected to be of use in the study of the effect of viscosity on e.g. the formation of stable shock fronts and on the determination of the onset and dynamics of Richtmyer–Meshkov instabilities in spherical implosions [2].

The combined use of FV and FE techniques is made possible by the introduction of suitable equivalence conditions that relate the FV metrics, i.e. cell volumes and integrated normals, to the FE integrals. Equivalence conditions relating the two schemes have been derived for Cartesian coordinates in two and three spatial dimensions [14,15] and for cylindrical coordinates in axially symmetric two-dimensional problems [7]. In both cited references, equivalence

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conditions are obtained by neglecting higher order FE contributions. Subsequently in [5], equivalence conditions for the cylindrical coordinates have been derived for the first time without introducing any approximation into the FE discrete expression of the divergence operator, and in [8] the difference between the consistent scheme and that violating the equivalence conditions have been examined for the case of one-dimensional problems in cylindrical and spherical coordinates. In the present paper the consistent formulation between FV and FE is extended to the case of spherical coordinates r, θ, ϕ, with r, θ and ϕ, respectively, radial, polar and azimuthal coordinate.

The present paper is structured as follows. In Section 2, the FE and FV schemes are briefly described for a scalar conservation law. Equivalence conditions are demonstrated in this case. The extension to the system of Euler equations for compressible flow is also sketched. In Section 3, numerical simulations are presented for the explosion and implosion problems in the spherical coordinates on the r–ϕ plane and are compared to one-dimensional simulations. In Section 4 final remarks and comments are given.

2. Finite volume/element method in spherical coordinates

In the present section, the finite element and finite volume discrete equations for an exemplary scalar conservation law in a three-dimensional spherical reference are given. The model equation reads

$$\frac{\partial u}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi} = 0,$$

where t is the time, r, θ and ϕ are the radial, polar and azimuthal coordinates, respectively, $u = u(r, \theta, \phi, t)$ is the scalar unknown and $\boldsymbol{f}^{\odot}(u) = (f_r, f_\theta, f_\phi)$ is the so-called flux function. A more compact form of the above equation is obtained by introducing the divergence operator in three-dimensional spherical coordinates, $\nabla^{\odot} \cdot (\cdot)$, as follows

$$\frac{\partial(u)}{\partial t} + \nabla^{\odot} \cdot \boldsymbol{f}^{\odot}(u) = 0, \tag{1}$$

with

$$\nabla^{\odot} \cdot \boldsymbol{f}^{\odot}(u) = \frac{2}{r} f_r + \frac{\partial f_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi}$$

Eq. (1) is first discretized in space by means of the FE and the FV method; equivalence conditions relating the two approaches are then derived. Finally, the numerical scheme is applied to the compressible Euler equations and time discretization is discussed.

2.1. Node-pair finite element discretization

The scalar conservation law (1) is first multiplied by the term $r\sin\theta$ to formally remove the singularity at the origin of the reference system, see [8]. The weak form of the resulting equation is obtained by multiplying the differential equation by test functions $\varphi \in V \subset H^1(\Omega)$ and integrating over the domain Ω as follows

$$\int_\Omega \varphi r\sin\theta \frac{\partial u}{\partial t} d\Omega^{\odot} + \int_\Omega \varphi r\sin\theta \nabla^{\odot} \cdot \boldsymbol{f}^{\odot}(u) d\Omega^{\odot},$$

to simplify the notation, the infinitesimal volume $d\Omega^{\odot} = r^2 \sin\theta dr d\theta d\phi$ will not be indicated in the integrals. An integration by parts gives

$$\int_\Omega \varphi r\sin\theta \frac{\partial u}{\partial t} = \int_\Omega \varphi \boldsymbol{f}^{\odot}(u) \cdot \nabla^{\odot}(r\sin\theta) + \int_\Omega r\sin\theta \boldsymbol{f}^{\odot}(u) \cdot \nabla^{\odot}\varphi - \oint_{\partial\Omega} r\sin\theta \varphi \boldsymbol{f}^{\odot}(u) \cdot \boldsymbol{n}^{\odot},$$

where $\partial\Omega$ is the domain boundary and $\boldsymbol{n}^{\odot} = n_r\hat{\boldsymbol{r}} + n_\theta\hat{\boldsymbol{\theta}} + n_\phi\hat{\boldsymbol{\phi}}$ is the outward normal versor to Ω.

The discrete form of the above equation is obtained by considering a finite dimensional space $V_h \subset H^1$ of Lagrangian test functions φ_h, to obtain

$$\int_{\Omega_i} \varphi_i r\sin\theta \frac{\partial u}{\partial t} = \int_{\Omega_i} \varphi_i \boldsymbol{f}^{\odot}(u) \cdot \nabla^{\odot}(r\sin\theta) + \int_{\Omega_i} r\sin\theta \boldsymbol{f}^{\odot}(u) \cdot \nabla^{\odot}\varphi_i - \int_{\partial\Omega_i^{\partial}} r\sin\theta\varphi_i \boldsymbol{f}^{\odot}(u) \cdot \boldsymbol{n}^{\odot}, \quad \forall i \in \mathcal{N}$$