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Original Article

Finite difference schemes satisfying an optimality condition for the unsteady heat equation

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Abstract

In this paper we present a formulation of a finite difference Crank–Nicolson scheme for the numerical solution of the unsteady heat equation in $2+1$ dimensions, a problem which has not been extensively studied when the spatial domain has an irregular shape. It is based on a second order difference scheme defined by an optimality condition, which has been developed to solve Poisson-like equations whose domains are approximated by structured convex grids over very irregular regions generated by the direct variational method. Numerical examples are presented and the results are very satisfactory.

Keywords: Numerical grid generation; Crank–Nicolson; Direct generation method; Finite difference method; Unsteady heat equation

1. Introduction

Several accurate finite difference methods for 2*D* rectangular domains for the numerical solution of the diffusion equation have been thoroughly studied in a number of papers [8,12,14–16,21], showing how to develop high order schemes which are appliable to a wide range of applications. However, despite its importance on the modelling of real-world domains, few schemes have been proposed when the spatial domain has an irregular shape.

In this paper, we address a simple formulation of a Crank–Nicolson scheme for the numerical solution of the unsteady heat equation in $2+1$ dimensions, which is based on the improved approach for the numerical solution of Poisson problems on irregular planar domains using a finite difference scheme defined by a least squares problem as discussed in [9].

Irregular plane regions can be accurately approximated by structured grids generated by variational methods. It must be recalled that a variational method to generate convex and smooth structured grids consists of minimizing an appropriate functional [5,11,13,18]. Area and harmonic functionals can be used for gridding a wide variety of

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simple connected domains in the plane [1,13,19], whose boundaries are closed polygonal Jordan curves with positive orientation.

For one of those domain boundaries, let us denote by m and n the number of "vertical" and "horizontal" numbers of points of the "sides". The boundary is the positively oriented polygonal curve γ of vertices

$$V = \{v_1, \cdots, v_{2(m+n-2)}\},$$

and it defines the typical domain Ω. A set

$$G = \{P_{i,j} | 1 \leq i \leq m, 1 \leq j \leq n\}$$

of points of the plane with the fixed boundary positions given by V is a structured grid with quadrilateral elements for Ω, of order $m \times n$.

A grid G is convex if and only if each one of the $(m-1)(n-1)$ quadrilaterals (or cells) $c_{i,j}$ of vertices $\{P_{i,j}, P_{i+1,j}, P_{i,j+1}, P_{i+1,j+1}\}$, $1 \leq i < m$, $1 \leq j < n$, is convex and non-degenerate.

The functional used to generate the structured grids of the numerical tests, as implemented in UNAMALLA [20], was the convex linear combination of the area functional $S_\tau(G)$ and length functional $L(G)$ with weight $\sigma = 0.5$ (See [2]). τ is an adaptive parameter which can be updated in such a way that in a finite number of updates the combined functional attains their minima within the set of convex grids for Ω if the latter is nonempty. For this paper, τ was set initially to $\tau = 1$ and updated after every 60 iterations of the grid generation process following the rule $\tau \leftarrow 1.1\tau$.

2. Finite difference approximation to $\nabla^2 u$

Finite difference schemes can be generalized by considering a finite set of nodes $p_0, p_1, p_2, \ldots, p_k$, for which it is required to find coefficients $\Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_k$ such that

$$\sum_{l=0}^{k} \Gamma_l u(p_l) \approx \frac{\partial^q u(p_0)}{\partial x^s \partial y^{q-s}}; \quad 0 \leq s \leq q, \quad q = 0, 1 \ldots \tag{1}$$

It is well known that the Γ_l values can be calculated easily in block-rectangular regions. However, despite the basic idea is quite simple, the use of Taylor's Theorem leads to more complicated schemes on irregular regions. Up to our knowledge, there are few efficient schemes for such kind of regions; among the papers in this direction are those of Castillo et al. [4,3,10], Shashkov [17], and Tinoco et al. [7]. In order to approximate the second order linear operator

$$Lu = \nabla^2 u = u_{xx} + u_{yy} \tag{2}$$

at the grid point p_0 by means of the difference scheme

$$L_0(p_0) \equiv \sum_{l=0}^{k} \Gamma_l(p_0, p_l) u(p_l) \tag{3}$$

we use the fact that the difference scheme L_0 is consistent if [6]

$$[Lu]_{p_0} - L_0(p_0) \to 0$$

when $p_1, \ldots p_k \to p_0$ [Celia, 1992].

In formula (3), the dependence of every Γ_l upon p_0 and p_l is explicitly written. However, for the sake of brevity, in the following paragraphs $\Gamma_l(p_0, p_l)$ will be denoted simply as Γ_l.