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Original article

Ternary shape-preserving subdivision schemes

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Abstract

We analyze the shape-preserving properties of ternary subdivision schemes generated by bell-shaped masks. We prove that any bell-shaped mask, satisfying the basic sum rules, gives rise to a convergent monotonicity preserving subdivision scheme, but convexity preservation is not guaranteed. We show that to reach convexity preservation the first order divided difference scheme needs to be bell-shaped, too. Finally, we show that ternary subdivision schemes associated with certain refinable functions with dilation 3 have shape-preserving properties of higher order.

Keywords: Subdivision scheme; Bell-shaped mask; Shape-preserving

1. Introduction

Shape-preserving approximations are used in the design of curves or surfaces to *predict or control their 'shape'* by the shape of the *control points*, i.e. the vertices of a given polygonal arc or polyhedral surface.

Efficient methods to construct shape-preserving approximations starting from an initial data sequence are *shape-preserving subdivision schemes*, i.e. schemes that preserve the *shape* – for instance, the monotonicity or the convexity – of the control points. There are several examples of binary monotonicity and convexity preserving subdivision schemes (see, for instance [1,7,9,17] and references therein) while the literature on ternary shape-preserving subdivision schemes is much smaller. Some first attempts in this direction are, for instance, the methods proposed in [2,3,13,16,19,20]. The use of dilation 3 gives more flexibility in the construction of subdivision schemes endowed with properties useful in applications. For instance, ternary subdivision schemes can improve the smoothness of the limit function while keeping a small support. Moreover, the convergence rate of ternary subdivision schemes is faster than the convergence rate of binary schemes since at each iteration the new sequence has three times as many points as the previous one so reducing the computational cost (cf. [13]).

Well-known examples of approximating shape-preserving subdivision schemes are the schemes generating spline curves [15]. Interestingly enough, for any arity they are associated with refinement masks that are *bell-shaped*.

Our goal is to analyze the shape-preserving properties of ternary subdivision schemes generated by bell-shaped refinement masks. Some results on monotonicity preservation of bell-shaped binary schemes can be found in [21]. Here, we want to analyze how the bell-shape property of a mask, as defined in Section 2, reflects on the shape-preserving

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properties of the associated subdivision scheme. In particular, the main result of the paper is that bell-shaped masks always generate monotonicity preserving subdivision schemes (see Section 3) while the bell-shape property is not sufficient to guarantee convexity preservation. In Section 4 we show which further assumptions on the masks are required to reach convexity preservation, too. Finally, in Section 5 we analyze a subdivision scheme family having shape-preserving properties of higher order and compare the behavior of these schemes with other ternary shape-preserving schemes from the literature.

2. Preliminaries

A (stationary) *ternary subdivision scheme* $\mathcal{S}_{\mathbf{a}}$ is described by the algorithm

$$\mathcal{S}_{\mathbf{a}}: \quad \begin{cases} \lambda^0 = \lambda = \{\lambda_\alpha\}_{\alpha\in\mathbb{Z}} \in \ell(\mathbb{Z}), \\ \lambda^{k+1} := S_{\mathbf{a}}\, \lambda^k, \quad k \geq 0, \end{cases} \tag{2.1}$$

where $S_{\mathbf{a}}$: $l(\mathbb{Z}) \to l(\mathbb{Z})$ is the *ternary subdivision operator* defined as

$$(S_{\mathbf{a}}\, \lambda)_\alpha = \sum_{\beta\in\mathbb{Z}} a_{\alpha-3\beta}\, \lambda_\beta, \quad \alpha \in \mathbb{Z}, \tag{2.2}$$

and the sequence $\mathbf{a} = \{a_\alpha \in \mathbb{R}\}_{\alpha\in\mathbb{Z}}$ is the *refinement mask.*

If the subdivision scheme is convergent, then there exists an uniformly continuous *limit function* f_λ, depending on the starting sequence λ, satisfying

$$\lim_{k\to\infty} \sup_{\alpha\in\mathbb{Z}} |\lambda_\alpha^k - f_\lambda(3^{-k}\alpha)| = 0, \tag{2.3}$$

with $f_\lambda \neq 0$ for at least a starting sequence. In the following, we will use the notation

$$f_\lambda = S_{\mathbf{a}}^\infty\, \lambda. \tag{2.4}$$

An equivalent definition of convergence requires the existence of the so-called *basic limit function* $\varphi_{\mathbf{a}}$ as the limit of the subdivision process when applied to the δ sequence, i.e.

$$\varphi_{\mathbf{a}} = S_{\mathbf{a}}^\infty\, \delta. \tag{2.5}$$

In fact, if the ternary subdivision scheme $\mathcal{S}_{\mathbf{a}}$ converges, the basic limit function is *refinable* with dilation 3, i.e. it satisfies the *refinement equation*

$$\varphi_{\mathbf{a}} = \sum_{\alpha\in\mathbb{Z}} a_\alpha\, \varphi_{\mathbf{a}}(3\cdot - \alpha), \tag{2.6}$$

and the limit function can be represented as

$$f_\lambda = \sum_{\alpha\in\mathbb{Z}} \lambda_\alpha\, \varphi_{\mathbf{a}}(\cdot - \alpha). \tag{2.7}$$

Most of the theory of the subdivision schemes consists in deriving the convergence conditions of the subdivision process and the properties of the basic limit function $\varphi_{\mathbf{a}}$ from the mask properties (see, for instance [5–7,12,15] and references therein).

We recall that a necessary condition for a ternary subdivision scheme to be convergent [12] is that its mask satisfies the *basic sum rules*

$$\sum_{\alpha\in\mathbb{Z}} a_{3\alpha+r} = 1, \quad r = 0, 1, 2. \tag{2.8}$$

The sum rules imply that the *mask symbol*, defined as the z-transform

$$a(z) = \sum_{\alpha\in\mathbb{Z}} a_\alpha\, z^\alpha, \tag{2.9}$$