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Original article

Center conditions and cyclicity for a family of cubic systems: Computer algebra approach

Brigita Ferčec [a], Adam Mahdi [b,c,∗]

[a] Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, SI-2000 Maribor, Slovenia
[b] Department of Mathematics, North Carolina State University, Campus Box 8205, Raleigh, NC 27695, USA
[c] Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Krakow, Poland

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Abstract

Using methods of computational algebra we obtain an upper bound for the cyclicity of a family of cubic systems. To that end we overcome the problem of nonradicality of the associated Bautin ideal by moving from the ring of polynomials to a coordinate ring. Finally, we also determine the number of limit cycles bifurcating from each component of the center variety.

Keywords: Cyclicity; Limit cycles; Center-focus problem

1. Introduction

We consider systems of ordinary differential equations on $\mathbb{R}^2$ of the form

$$\begin{aligned} \dot{u} &= \lambda u - v + \sum_{j+k=2}^{N} A_{j,k} u^j v^k = P(u, v), \\ \dot{v} &= u + \lambda v + \sum_{j+k=2}^{N} B_{j,k} u^j v^k = Q(u, v), \end{aligned} \tag{1}$$

where λ is arbitrarily close to zero (possibly zero). The *degree* of system (1) is $N = \max\{\deg P, \deg Q\}$. Depending on nonlinear terms the origin of system (1) is either a *center* (every orbit is an oval surrounding the origin), or a *focus* (every trajectory spirals towards or away from the origin). The problem of distinguishing between a center and a focus is called the *center* or the *center-focus* problem, for more details, see e.g. [20].

For system (1) denote by (λ, A, B) the set of its parameters λ, $A_{j,k}$ and $B_{j,k}$, and by $E(\lambda, A, B)$ the associated space of parameters. Let also $n_{(\lambda,A,B),\varepsilon}$ denote the number of limit cycles of system (1) that lie wholly within an ε-neighborhood of the origin. We define the key concept of this article, namely the cyclicity of a singular point.

∗ Corresponding author. Tel.: +1 9195132302.
E-mail addresses: brigita.fercec@gmail.com (B. Ferčec), amahdi@ncsu.edu (A. Mahdi).

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We say that the singularity at the origin for system (1) with fixed coefficients $(\lambda^*, A^*, B^*) \in E(\lambda, A, B)$ has *cyclicity k* with respect to the space $E(\lambda, A, B)$ if there exist positive constants δ_0 and ε_0 such that for every pair δ and ε satisfying $0 < \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0$

$$\max\{n_{(\lambda,A,B),\varepsilon} : |(\lambda, A, B) - (\lambda^*, A^*, B^*)| < \delta\} = k.$$

The problem of cyclicity of a center or a focus of a system of the form (1), which we always assume to be located at the origin, is also known as the *local 16th Hilbert problem* [11].

The concept of cyclicity was introduced by Bautin in his seminal paper [1], where he showed that the cyclicity of focus or center in quadratic systems is three. The cyclicity of the quadratic system has been studied by other methods in [14,24]. The cyclicity for some Liénard systems has been studied recently in [22,25]; for some systems of high degrees in [18]; and the relationship between the cyclicity and the center problem in [12].

The cyclicity for systems with quadratic and cubic homogeneous nonlinearities can be established easily using algorithms of computational algebra because the Bautin ideal for these systems is radical. The problem becomes more difficult if the Bautin ideal is not radical (which appears to be the generic case). An approach to finding the cyclicity in the case of nonradical Bautin ideal has recently been proposed in [17].

In this paper we further generalize the method of [17] and apply it to the study of the cyclicity problem for a family of real cubic systems whose expression in the complex form is

$$\dot{x} = \lambda x + ix(1 - a_{10}x - a_{20}x^2 - a_{11}x\bar{x} - a_{02}\bar{x}^2), \tag{2}$$

where $x = u + iv$ (the connection between system (1) and (2) is explained in detail in Section 2). The motivation for studying system (2) is that it is one of the very few 5-parameter cubic systems where the computation of the primary decomposition of the Bautin ideal is feasible (because of computational complexity it is extremely difficult to treat 6-parameter cubic systems with modern tools of computational algebra even using very powerful computers). The center problem for system (2) has been solved in [3].

If we add to (2) the complex conjugate equation and consider $\bar{x}$ as a new unknown function y, and $\bar{a}_{ij}$ as new parameters b_{ji} we obtain the associated complex system

$$\begin{aligned} \dot{x} &= \lambda x + ix(1 - a_{10}x - a_{20}x^2 - a_{11}xy - a_{02}y^2), \\ \dot{y} &= \lambda y - iy(1 - b_{01}y - b_{02}y^2 - b_{11}xy - b_{20}x^2), \end{aligned} \tag{3}$$

which is also called the *Lotka-Volterra* system (see [5]). We shall study in detail the structure of the Bautin ideal of system (3) as well as its variety. This will be used to derive bounds for the cyclicity of the origin of the *real* system (2). We will see in Section 4 that the proposed method gives an estimation for the cyclicity of an elementary center or focus for "almost all" points of the center variety. If with this approach one can algorithmically obtain a bound for the cyclicity for "almost all" points of the center variety of any polynomial family, then it can be considered as a solution to the cyclicity problem for an elementary center or focus. However to confirm or reject this hypothesis more studies are needed.

The article is organized as follows. In Section 2 we discuss a general approach to studying cyclicity of polynomial systems. In Section 3 we characterize the existence of the local analytic first integral of system (3). This is a preliminary result needed in Section 4 in order to solve the cyclicity problem of system (2). Finally, in Section 5 we estimate the cyclicity of each component of the center variety of system (2).

2. Bautin ideal and cyclicity

In this section we briefly review an approach for studying the cyclicity problem. We also discuss a method for bounding the cyclicity of a singular point by moving from the ring of polynomials to a new ring, in which the Bautin ideal becomes radical or has a simple structure.