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Sliding mode control: A survey with applications in math

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Abstract

The paper presents a brief survey on Variable Structure Control Systems with Sliding Modes. Starting from a general case of sliding modes in dynamical systems with discontinuous right-hand side, classic approaches to sliding mode control systems are considered and some basic results about the control of uncertain systems are given. Then, Higher-Order Sliding Modes are presented as a tool to remove discontinuity from the control action, to deal with higher relative degree systems and to improve the accuracy of the real sliding mode behavior when the discrete time implementation is considered.

Finally, three applications of the sliding mode control theory to applied math problems are presented: the numerical solution of constrained ODEs, the real-time differentiation, and the problem of finding the zeroes of nonlinear algebraic systems. The first is an almost straightforward application of the sliding mode control theory, while the last two are accomplished by computing the solution of properly defined dynamical systems. Some simulations are reported to clarify the approach.

Variable structure systems; Sliding modes; Differentiation; Constrained ODE; Zero finding

1. Introduction

Nonlinear dynamical systems have been considered an interesting topic of investigation because of the possible follow up of the research results. Actually, real systems are always nonlinear and considering their linear approximation can impose too strict requirements on their working range or give unfeasible results. Furthermore, nonlinear systems can even provide better performance than linear ones, and often some nonlinear behavior is intentionally introduced in feedback control systems [57,92,59].

Among nonlinear control systems, switching control systems are quite interesting since they are very simple to implement [96,110] or even the optimal solution to some control problems [2]. Switching dynamical systems originate interesting mathematical problems since they are characterized by ODE (Ordinary Differential Equations) with discontinuous right-hand side and the usual definitions and existence conditions for the solution of a ODE are no longer valid; therefore a proper extension of classic differential equations theory has to be taken into account [46].

Switching systems are characterized by changes in the system dynamics, associated to different sets in the state space [26]. These sets are separated each other by a border, often named as *the guard* in the hybrid systems literature [1], and it can happen that the vector fields across the border are directed toward the border itself. In this case a stable *sliding mode* arises and the border between the sets in the state space, defining different vector fields, is usually referred to as the *sliding surface* [98,38].

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In the case of existence of a stable sliding mode, the sliding surface is an invariant set in the state space and under proper conditions the state trajectory is independent from the original systems dynamics [36], i.e., the constrained motion presents a semi-group property [101].

Such an invariance property, with respect to matching uncertainties, of sliding mode systems interested control engineers who considered the opportunity of introducing switchings in the feedback intentionally, so that the closed loop control system has the desired performance, in spite of system uncertainties and external disturbance satisfying the matching condition [97,32,106].

In the last two decades a number of books [98,38][5], collections [112,55,107,109,83,40,21], special issues [8,74,53,22], and articles on the topic have been published.

Most of contributions are related to control systems where sliding mode system theory is used for the controller design [105,37,85], possibly in combination with other techniques [9,27,84,80,58], while quite few applications to mathematical problems have been presented [60,77]. Nevertheless, sliding mode theory can have interesting applications in solving ill-posed mathematical problems. Actually, if the variable structure system is represented by an equivalent differential inclusion, sliding mode control can be interpreted as a way of choosing a peculiar function from a set of possible solution (i.e., the solution of a differential inclusion that satisfies a constraint), or as a generator for the *internal model* [48].

This feature is exploited in partial or full system inversion for state observation [91,47,94,31] or unknown input estimation [86] and fault detection and identification [39]. Taking into account the control theory, and in particular the sliding mode approach, the system inversion problem, that in general implies the ill-posed problem of differentiating a known signal, is regularized by considering a proper dynamic feedback system (one or more integrators) to be stabilized to zero.

The aim of this paper is to give the reader an overview of the sliding mode control theory, showing the developments from classical sliding modes to the higher-order sliding modes approach, and to present some applications to mathematical problems such as constrained ODE, differentiation, and finding zeroes of algebraic nonlinear systems.

The paper is organized as follows. First, the classic sliding mode control approach is presented in Section 2. Then higher-order sliding modes are discussed with some detail in Section 3. Since ideal sliding modes need infinite frequency switchings that are not possible in real systems, or when considering numerical algorithms, the effect of discretization is discussed in Section 4. The considered applications are presented and discussed in Section 5, and finally some conclusions are drawn.

1.1. Notation and definitions

In this section some notations that will be used in the following treatment are given to the readers' convenience, as well as few definitions.

Notation 1. Let $\mathbf{z} = [z_1, z_2, \ldots, z_k]^T \in \mathbb{R}^k$ be a vector of arbitrary dimension. Define the vectors sign($\mathbf{z}$), $|\mathbf{z}|$ and $\mathbf{z}^{p/q}$, with $p > q$ odd natural numbers, as follows:

$$\mathrm{sign}(\mathbf{z}) = \left[\mathrm{sign}(z_1),\ \mathrm{sign}(z_2), \ldots, \mathrm{sign}(z_k)\right]^T,$$
$$|\mathbf{z}| = [|z_1|, |z_2|, \ldots, |z_k|]^T,$$
$$\mathbf{z}^{p/q} = \left[z_1^{p/q}, z_2^{p/q}, \ldots, z_k^{p/q}\right]^T.$$

Notation 2. Consider a scalar function $h(\mathbf{z})$, $h : \mathbb{R}^k \rightarrow \mathbb{R}$. The gradient vector is denoted as follows:

$$\nabla h = \frac{\partial h}{\partial \mathbf{z}} = \left[\frac{\partial h}{\partial z_1}, \ldots, \frac{\partial h}{\partial z_k}\right].$$