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Continuation of oscillatory patterns in a road traffic model of three cars

Lubor Buřič [a], Vladimír Janovský [b,*]

[a] *Department of Mathematics, Institute of Chemical Technology, Technická 5, 16628 Prague 6, Czech Republic*
[b] *Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 18675 Prague 8, Czech Republic*

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Abstract

We investigate microscopic models of the road traffic. In particular, we consider the car-following model for a single-line traffic flow of N identical cars on a circular road. The classical differentiable model breaks down at the time instant when two cars collide. Nevertheless, the natural action of a driver would be to overtake the slower car. In our previous work, we proposed the model which simulates overtaking. We observed a large variety of oscillatory solutions (*oscillatory patterns*) of the model. In the present contribution, we assume $N=3$ i.e., three cars on the route, and formulate our model as a particular piecewise smooth (PWS) dynamical system. We define generic oscillatory patterns as invariant objects of the flow defined by this PWS system. We use the standard software to continue the patterns with respect to a parameter (the length of the route). In this respect, we also investigate bifurcation phenomena.

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1. Introduction

The aim of this contribution is to present some new results of the analysis of a road traffic model with overtaking. For a general reference to the modeling of traffic flows, see [15]. We consider a particular *microscopic* model of the traffic flow on a circular road of length L in the following form

$$x_i' = y_i, \tag{1a}$$

$$y_i' = \frac{1}{\tau}\left[V(x_{i+1} - x_i) - y_i\right], \quad x_{N+1} = x_1 + L, \tag{1b}$$

$i = 1, \ldots, N$, where N is the number of cars, see [3]. Each pair $(x_i, y_i) \in \mathbb{R}^2$ defines the position and the velocity of the i th car. The model depends on the choice of a nondecreasing function V which is called the *optimal velocity function*.

* Corresponding author.
E-mail addresses: lubor.buric@gmail.com (L. Buřič), janovsky@karlin.mff.cuni.cz (V. Janovský).

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Note that the variable x_i is personified as the driver of the i th car. The choice of the function V imposes the same driving law for each of the drivers. In this paper, we consider the hyperbolic optimal velocity function

$$V(r) = V^{\max}\frac{\tanh(a(r-1)) + \tanh a}{1 + \tanh a}.$$

Hence, the model (1) depends on four positive parameters L, τ, $V^{\max}$, a. It can be classified as a *follow-the-leader* model.

Given an initial condition $(x^0, y^0) \in \mathbb{R}^N \times \mathbb{R}^N$, system (1) defines a flow on $\mathbb{R}^N \times \mathbb{R}^N$. Without loss of generality, we may order x^0 as

$$s \leq x_1^0 \leq x_2^0 \leq \cdots \leq x_N^0 \leq s + L,$$

where $s \in \mathbb{R}$ is an arbitrary phase shift. The quantity

$$h_i \equiv x_{i+1} - x_i, \quad i = 1, \ldots, N, \tag{2}$$

is called the *headway* of the i th car. It is the distance between the i th car and its *leader* (the car ahead).

Note that driving laws may be much more complicated e.g., defined by a system of delay differential equations, see [2,19]. For a recent development in modelling of *the intelligent driver*, see e.g. [17]. It is well known that optimal velocity models could break down, see e.g. [12,19].

The particular model (1) may collapse at a finite time due to the collision of cars. More precisely, the collision occurs at time $t_E > 0$ for which there exists $k \in \{1, 2, \ldots, N\}$ such that

$$h_k(t_E) = x_{k+1}(t_E) - x_k(t_E) = 0, \quad y_k(t_E) > y_{k+1}(t_E).$$

In [5], we suggested that this event can be interpreted in the following way: at time t_E, the car No. k is about to overtake (pass) the car No. $(k+1)$. Consequently, we proposed and justified the model of a *traffic flow with overtaking*. In general, this model is an *event-driven* model, see [10] for an overview. It is a natural extension of model (1). The objective is to study periodic solutions which we called *oscillatory patterns* [5]. Oscillatory patterns exhibit spatial–temporal symmetries. Note that a different approach to model the overtaking is presented in [16].

Transforming the state variables $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, we can reformulate the traffic flow with overtaking as a *piecewise smooth (PWS) dynamical system*, see [10] for a general reference. This reformulation can be done for an arbitrary $N \geq 3$, see [6]. The structure of PWS systems is well understood and we can exploit it in order to analyze our particular problem.

In what follows, we consider the case $N = 3$ i.e., three cars on the route. In [8,7], we explored the possibility to *continue* the oscillatory patterns with respect to the parameter L. Note that an oscillatory pattern is an infinite-dimensional *invariant* object. It is defined as a boundary value problem with special boundary value conditions: we speak of *Defining Equations* of the object. Moreover, this object can be numerically approximated. In [7], we identified five basic oscillatory patterns. We presented results concerning the numerical continuation of these patterns. Nevertheless, we missed to give the complete list of Defining Equations which are rather complicated. In [8], as a case study, we gave defining conditions for the simplest one of these patterns (the three-legged dog pattern).

The paper is organized as follow: in the next section, we introduce a more convenient topology of the state space. We give a modified definition of the PWS dynamical system which models the overtaking. This definition is equivalent to [8,7]. In Section 3, we present a list of Defining Equations for the basic oscillatory patterns of the traffic flow. We consider numerical continuation of Defining Equations with respect to parameter L. The aim is to analyze bifurcation phenomena. We give illustrative examples.

2. Road traffic model with overtaking

Let us underline the assumption $N = 3$ made in Introduction. We introduce a new state variable which is linked to the actual position $x \in \mathbb{R}^3$ on the route. For a motivation, see [8,7]:

Definition 1. The quantity

$$h_{i,j} = \frac{x_j - x_i}{L} \quad \text{as} \quad i < j \quad \text{and} \quad h_{i,j} = 1 - h_{j,i} \quad \text{as} \quad i > j \tag{3}$$

is called the *normalized gap* between car No. i and car No. j.