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The robustness of a Nash equilibrium simulation model: Game-theoretic approach using variable metric projection method

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Abstract

This paper proposes a Nash equilibrium model that applies continuous time replicator dynamics to the analysis of oligopoly markets. The robustness of the proposed simple Nash equilibrium model under the simultaneous constraints of allocation of product and market share using a simulation method to derive an optimal solution for production decisions by rival firms in oligopoly markets is tested by changing profit and cost function parameters, as well as the initial production values and market shares of the firms examined in this study. The effects of differences in conjectural variation and initial allocation of market share on the convergent values are considered, particularly in the case of corner solutions. This approach facilitates the understanding of the robustness of attaining equilibrium in an oligopoly market.

Keywords: Nash equilibrium; Replicator dynamics; Oligopoly

1. Introduction

Oligopoly markets prevail in developed countries in both the industrial and service sectors. See, for example, Dixit [3], which examines the U.S. automobile industry, and Klepper [4], which examines the airline industry. Competition and conjecture play an important role in improving profits in oligopoly firms. The decision-making process and solution under such circumstances has been studied in game theory. In the game-theoretic approach, one usually focuses on two problems: (1) What is a rational strategy regarding rational allocation in a market among agents that compete with each other in an oligopoly market? (2) How do they change their behavior in order to reach the rational allocation? The former is a static problem to obtain a desirable strategy under the conditions of competition and conjecture, while the latter is a dynamic problem to find a continuous modification that moves toward the desirable strategy.

In this study, we propose a simple Nash equilibrium model and use a simulation method to derive an optimal solution for production decisions by rival firms in oligopoly markets. Aiyoshi and Maki [1] proposed a Nash equilibrium model that applies continuous time replicator dynamics to the analysis of oligopoly markets. Here we consider the game problem of allocating both product and market share, namely resource allocation within the firm to produce their

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products and allocation of market share among firms for each product. We will propose a replicator dynamic equation to change strategies in order to obtain the Nash equilibrium solutions. The stable solutions for the replicator dynamic equation are the Nash equilibria. The dynamic process of the game-theoretic approach using the variable metric projection method obtains the Nash equilibria with the double allocation constraints.

Before conducting empirical analysis using observation on oligopoly markets in the real world, we have to check the robustness of the Nash equilibrium model by changing parameters of firms' profit and cost functions. This process is necessary to conduct forecasting and simulation using real-world observations after estimating the model.

This paper is organized as follows. Section 2 provides a general explanation of the double constraint interference model, which concentrates on the double allocation problem of production capacity and market share. Section 3 introduces a normalized Nash equilibrium solution for the profit maximization of players' functions modeled in the double constraint interference model. Section 4 describes the application of numerical methods to the Nash equilibrium solution.

A simulation model is proposed in Section 5 and the results are reported. We focus on the effect of conjectural variation in the model. Section 5.1 specifies the model and obtains the benchmark solution in the case of a three-person game with three products. Section 5.2 shows how convergent values are affected by changes in the conjectural variations. When the values of the conjectural variations are relatively small, the model converges to an interior solution. On the other hand, when the values of the conjectural variation are larger, the model obtains a corner solution, namely some particular agent concentrates on the production of a single product. As an extension of Aiyoshi et al. [2], this section considers the stability of the corner solutions. When a corner solution is unstable, it is not a Nash equilibrium solution, while when a corner solution is stable, it is a Nash equilibrium solution. In the present analysis, we obtain the stable Nash equilibrium solutions by applying replicator dynamics, the process of which is mathematically specified in Sections 3 and 4. The stability is also checked by another method of dynamic processes, called the penalty method. Section 5.3 presents the simulation of changes in the convergent values due to changes of the initial allocation.

Section 6 presents the conclusions.

2. Non-cooperative Nash equilibrium model and resource allocation

Consider a continuous game problem with P players and N strategy variables, governed by duplicate simplex constraints. The pth player's strategy variables are $\boldsymbol{x}^p = (x_1^p, \cdots, x_N^p) \in R^N$ and $i = 1, \cdots, N,\ \ p = 1, \cdots, P$. The variable matrix X that contains all variables is

$$X = (\boldsymbol{x}^1, \cdots, \boldsymbol{x}^P) = \begin{pmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_N \end{pmatrix} = \begin{pmatrix} x_1^1 & \cdots & x_1^P \\ \vdots & \ddots & \vdots \\ x_N^1 & \cdots & x_N^P \end{pmatrix}, \tag{1}$$

where $\boldsymbol{x}^p$, $p = 1,\cdots,P$ are column vectors and $\boldsymbol{x}_i$, $i = 1,\cdots,N$ are row vectors. Let the pth player's profit function be $E^p(X)$. An unconstrained game problem is formulated as

$$\max_{\boldsymbol{x}^p} E^p(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^p, \cdots \boldsymbol{x}^P),\ \ p = 1, \cdots, P, \tag{2}$$

where $\boldsymbol{x}^p$ is the pth player's only known variable and the other players' variables $(\boldsymbol{x}^1,\cdots,\boldsymbol{x}^{p-1},\boldsymbol{x}^{p+1},\cdots,\boldsymbol{x}^P)$ are unknown parameters. Consider the game problem constrained by the simultaneous allocation of products and market share as

$$\max_{\boldsymbol{x}^p} E^p(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^p, \cdots \boldsymbol{x}^P) \tag{3a}$$

$$\text{subj. to } \sum\nolimits_{i=1}^{N} x_i^p = a^p,\ \ p = 1, \cdots, P, \sum\nolimits_{p=1}^{P} x_i^p = b_i,\ \ i = 1, \cdots, N, \tag{3b}$$

$$x_i^p \geq 0,\ \ i = 1, \cdots, N,\ \ p = 1, \cdots, P, \tag{3c}$$

where by definition,

$$\sum\nolimits_{p=1}^{P} a^p = \sum\nolimits_{i=1}^{N} b_i. \tag{4}$$