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# PCF self-similar sets and fractal interpolation

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**Abstract**

The aim of this paper is to show, using some of Barnsley's ideas, how it is possible to generalize a fractal interpolation problem to certain post critically finite (PCF) compact sets in $\mathbb{R}^n$. We use harmonic functions to solve this fractal interpolation problem.


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## 1. Introduction

A long-standing question in mathematical analysis on fractals is the existence of a "Laplace operator" on a given self-similar set. In the last decades several approaches have been developed from both probabilistic and analytic viewpoints. The analytic approach goes back to Kigami (see [11,10]), where a general theory of analysis on post-critically finite (PCF) self-similar sets is developed when assuming the existence of the so-called self-similar harmonic structure or, equivalently, a self-similar Laplace operator.

On the other hand, Barnsley introduced in [1] a fractal interpolation method based on Hutchinson [9] and using iterated function system (IFS) theory. A function $f : \mathbb{I} \to \mathbb{R}$, where $\mathbb{I} := [0, 1]$ is a real closed interval, is named by Barnsley as a *fractal function* if its graph is a fractal set. Furthermore, fractals can be realized as the attractor of an IFS.

This method has been used in the unit interval $\mathbb{I}$ to generalize Hermite functions by fractal interpolation, and to study spline fractal interpolation functions (see [13,14]).

In other direction, considering the polynomials of degree 1 as classical harmonic functions on $\mathbb{I}$ and replacing them on the Sierpinski gasket (SG) (respectively, on the Koch Curve (KC)) by harmonic functions of fractal analysis, an analog to the Barnsley fractal interpolation result for SG (respectively, KC) is obtained in [5] (respectively [15]).

We find an essential difference between the result by Barnsley and the others in [5,15]. For real intervals, the graph of the function is possible to be obtained as the attractor of an IFS, but in the cases of Sierpinski gasket and Koch Curve, it is impossible to ensure that the corresponding graph of the interpolating function is the attractor of an IFS. In this paper we state a generalized fractal interpolation problem in PCF that includes the preceding cases. We start

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to study the problem with a family of bounded functions. Afterwards, we describe sufficient conditions to obtain the solution of the interpolation problem via an IFS. The last result generalizes to PCF the ones obtained in [5,15]. The benefit of using the set of harmonic functions of a PCF in interpolation is that they ensure we can find exactly one of them under the stated assumptions, however for other kinds of families, for instance polynomials, there exist cases where neither the uniqueness nor the existence of such solution are satisfied.

The paper is organized as follows: in Section 2 we review the notations and preliminary facts of the objects we investigate. Section 3 contains the main results of this paper. We state the conditions to solve a generalized Barnsley interpolation problem on a compact $K \subset \mathbb{R}^n$, and we generalize the results that are already known for real closed intervals (Theorem 2.1). This study is applied to PCF self-similar structures (Theorem 3.6), and to harmonic structures (Theorem 3.12) in the sense of Kigami (see [12, Chapter 6]). The results we obtain can be applied to the Lagrangian interpolation problem (Corollary 3.8). We give some classic examples; other examples are new. They show how our study works.

## 2. Notions and definitions

We review in this section, for the reader's convenience, the main definitions and properties concerning the IFS theory, fractal interpolation, and self-similar and harmonic structures, that we use below. For a more detailed account, see [12, Section 1.3] and [2, 6].

### 2.1. Self-similar set generated by a finite system of similitudes

Within fractal geometry, the method of iterated function systems (see [3,9]) is a relatively easy way to generate fractal sets.

Let $(X, d)$ be a complete metric space. A function $f: X \to X$ is called a *similitude* (or a *contracting similarity*) of ratio $\lambda$ if there is $\lambda \in ]0, 1[$, such that $d(f(x), f(y)) = \lambda d(x, y)$, for all $x, y \in X$. A similitude transforms subsets of $X$ into geometrically similar sets. An *iterated function system*, or IFS for short, is a collection of a complete metric space $(X, d)$ together with a finite set of similitudes $F_i: X \to X$, $i = 1, 2, \ldots, N$. It is often convenient to write an IFS as $\{X: F_1, F_2, \ldots, F_N\}$. For an introduction to representation techniques of many fractals by function systems, see [7,8], and for a study of the speed of convergence of the "approximate fractal" and the limit fractal set (in terms of preselected parameters) you can see [6].

The following result ensures the uniqueness and existence of self-similar sets (see, for example [9]): Let $\{X: F_1, F_2, \ldots, F_N\}$ be an IFS. Then there exists a unique non-empty compact subset $K$ of $X$ that satisfies

$$K = \bigcup_{i=1}^{N} F_i(K).$$

The compact $K$ is called the *self-similar set* or the *attractor set with respect to* the system $\{X: F_1, F_2, \ldots, F_N\}$.

For a finite family $\{F_1, F_2, \ldots, F_N\}$ of similitudes acting on $X$, we write $|F_i(x) - F_i(y)| = \lambda_i |x - y|$ for numbers $\lambda_i \in ]0, 1[$, $i = 1, \ldots, N$.

### 2.2. Fractal interpolation functions

Suppose that a set of data points

$$\Gamma := \{(x_i, u_i) \in \mathbb{R}^2 : i = 0, 1, 2, \ldots, N; N \geq 2\}$$

is given, where $x_0 < x_1 < \cdots < x_i < x_{i-1} < \cdots < x_N$. An *interpolation function* corresponding to this set of data is a continuous function $h : [x_0, x_N] \to \mathbb{R}$ such that $h(x_i) = u_i$ for every $i = 0, 1, 2, \ldots, N$. We say that function $h$ interpolates the data, and that the graph of $h$ (denoted by $G(h)$), passes through interpolation points $(x_i, u_i)$. Barnsley [2, Chapter 6] explains how one can construct an IFS in $\mathbb{R}^2$ such that its attractor is $G(h)$, with $h$ interpolating the data.

We note $\langle N \rangle := \{1, 2, 3, \ldots, N\}$. Let $F_i: [x_0, x_N] \to [x_{i-1}, x_i]$ be contractive homeomorphisms defined by $F_i(x) = m_i x + n_i$, where $m_i = (x_{i-1} - x_i)/(x_0 - x_N)$ and $n_i = (x_0 x_i - x_N x_{i-1})/(x_0 - x_N)$ for $i \in \langle N \rangle$. Let $M_i : [x_0, x_N] \to \mathbb{R}$