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Instability in supercritical nonlinear wave equations: Theoretical results and symplectic integration

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Abstract

Nonlinear wave evolutions involve a dynamical balance between linear dispersive spreading of the waves and nonlinear self-interaction of the waves. In sub-critical settings, the dispersive spreading is stronger and therefore solutions are expected to exist globally in time. We show that in the supercritical case, the nonlinear self-interaction of the waves is much stronger. This leads to some sort of instability of the waves. The proofs are based on the construction of high frequency approximate solutions. Preliminary numerical simulations that support these theoretical results are also reported.

Keywords: Nonlinear wave equations; Supercritical equations; Ill-posedness

1. Introduction

In this paper, we discuss some *ill-posedness* issues (mainly, established in [1,5]) for solutions of the *supercritical* non-linear wave equation

$$\partial_t^2 u - \Delta u + f(u) = 0 \ , \tag{1}$$

where $u = u(t, x) : \mathbb{R}^{1+d} \to \mathbb{R}$. We assume that the initial data

$$\gamma := \partial u_{|t=0} = (\nabla u, \partial_t u)_{|t=0} \tag{2}$$

is in the *homogeneous* Sobolev space $\dot{H}^{s-1}$ endowed with the norm

$$\|\gamma\|^2_{\dot{H}^{s-1}} := \int_{\mathbb{R}^d} |\xi|^{2(s-1)} |\hat{\gamma}(\xi)|^2 \ d\xi.$$

The nonlinear interaction f satisfies $f(0) = 0$ and it is supposed of the form $f = \partial V/\partial \bar{z}$ with a *potential* $V \in C^{\infty}(\mathbb{C}; \mathbb{R})$. This assumption on f formally guarantees the *conservation of the energy*

$$E(u(t)) := \int_{\mathbb{R}^d} |\partial u|^2 \, dx + \int_{\mathbb{R}^d} V(u) \, dx \ = E(u(0)). \tag{3}$$

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First, we recall our definition of *well-posedness* of the Cauchy problem.

Definition. We say that the Cauchy problem (1) and (2) is *locally well-posed* in $\dot{H}^s(\mathbb{R}^d)$ if for any bounded subset $\mathcal{B} \subset \dot{H}^{s-1}$, there exists a time $T = T(\mathcal{B}) > 0$ such that for every $\gamma \in \mathcal{B}$ there exists a (distributional) solution u to (1);

$$\partial u \in \mathcal{C}([-T, T]; \dot{H}^{s-1}),$$

and such that the solution map $\gamma \mapsto \partial u$ is uniformly continuous from $\mathcal{B}$ to $\mathcal{C}([-T, T]; \dot{H}^{s-1})$. Moreover, there is an additional space X in which u lies, such that u is the unique solution to the Cauchy problem in $\mathcal{C}([-T, T]; \dot{H}^s) \cap X$.

Second, we define the notion of *supercriticality*.

Definition. Let $s \in [0, d/2)$. The Cauchy problem (1) and (2) is said $\dot{H}^s$-*supercritical* if

$$(\mathbf{H}_{\mathrm{s,d}}) \qquad \frac{V(u)}{|u|^{(4/(d-2s))+2}} \uparrow +\infty \quad \text{as} \quad u \to +\infty.$$

To better illustrate the above condition let's first refer to the "model" case when the nonlinearity is given by

$$f(u) = |u|^{2\sigma} u. \tag{4}$$

In such a case, solutions to the nonlinear wave equation enjoy a scaling property. Indeed, if u solves (1) on $(-T^*, T^*)$ with initial data $\partial u_{|t=0} \in \dot{H}^{s-1}$, then for any $\lambda > 0$, the function $u^\lambda : (-T^*\lambda^2, T^*\lambda^2) \times \mathbb{R}^d$ defined by $u^\lambda(t, x) := \lambda^{-1/\sigma} u(\lambda^{-2}t, \lambda^{-2}x)$ also does. Moreover, the norm of the Sobolev space $\dot{H}^{s_c}$ with $s_c = (d/2) - (1/\sigma)$ is also invariant under the dilation $u \mapsto u^\lambda$. It turns out that this space is relevant in the theory of the initial value problem (1) and (2). Obviously, when f is given by (4), the *supercriticality* condition $(\mathbf{H}_{s,d})$ is equivalent to $s < s_c$.

Finally, recall that in the power nonlinearity case, problem (1) and (2) is locally well-posed for $s > s_c$ with an existence time interval depending only upon $\|\gamma\|_{\dot{H}^{s-1}}$, see [3]. It is also locally well-posed for $s = s_c$ with an existence time interval depending upon the solution to the linear wave equation $U_0(t)\gamma$, and is ill-posed for $s < s_c$ (see the reference work of [2], and the thorough paper of [6] concerning the loss of regularity of $\dot{H}^s$- *supercritical* waves). Based on this complete trichotomy, it is natural to refer to $\dot{H}^{s_c}$ as the *critical* regularity for (1). For more details, we refer to Tao's book [8] and the references therein. The case of $\dot{H}^1$-*supercritical* problems is of particular interest since in such a case, in addition, solutions enjoy the conservation of the energy (3). Moreover in this case, the definition of $\dot{H}^1$-*supercritical* problems $(H_{s,d})$ has to be extended when the space dimension $d = 2$. Indeed, energy-critical nonlinearities seem to be of exponential type.[1]

$$(\mathbf{H}_{1,2}) \qquad \begin{cases} (a) \quad \dfrac{\log(V(u))}{u^2} \quad \text{increases to} + \infty \\ (b) \quad liminf \dfrac{\log(V(u))}{u^2} = 4\pi \ \text{ and } \ E > 1. \end{cases}$$

The goal of this paper is to present some extensions of the results in [2,6] to $\dot{H}^s$-*supercritical* problems with more general nonlinearities given by $(H_{s,d})$. Similar results are also proved for the energy-*supercritical* problems in two space dimension. The last section is devoted to numerical simulations that show the change of dynamics through the different regimes. We refer to [1,5] for analogous results for nonlinear *Schrödinger* equations.

2. Ill posedness of $\dot{H}^s$ supercritical waves

The main result in this case is given in [1].

[1] In fact, the critical nonlinearity is of exponential type as shown in [4,5].