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Orbital stability of negative solitary waves

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Abstract

The generalized regularized long-wave equation admits a family of negative solitary waves. The stability of these waves is investigated by numerical simulation using a spectral discretization.

Keywords: Solitary waves; Orbital stability; Spectral discretization

1. Introduction

We will investigate the stability of the negative solitary-wave solutions of the generalized regularized long-wave equation

$$u_t + u_x + (u^p)_x - u_{xxt} = 0, \tag{1.1}$$

where $p \geq 2$ is a positive integer. Note that for the values $p=2$ and $p=3$ in this equation, which is also called the generalized BBM equation, is used to model the propagation of small-amplitude surface waves on a fluid running in a long narrow channel [6,14,17]. Eq. (1.1) admits solitary-wave solutions of the form $u(x, t) = \Phi_c(x - ct)$. Indeed, when this ansatz is substituted into (1.1), there appears

$$-c\Phi_c + \Phi_c + c\Phi_c'' + \Phi_c^p = 0, \tag{1.2}$$

and it is an elementary check that a particular solution of (1.2) is given by

$$\Phi_c(x - ct) = A\,\mathrm{sech}^\sigma[K(x + x_0 - ct)], \tag{1.3}$$

where $\sigma = 2/(p-1)$, $K = (p-1)/2\sqrt{(c-1)/c}$, and

$$A = \left[\frac{(p+1)(c-1)}{2}\right]^{1/(p-1)}. \tag{1.4}$$

These solutions are strictly positive progressive waves which propagate to the right without changing their profile over time. As can be seen from the expression (1.3), solitary waves with positive propagation velocity are defined only when $c > 1$. For $p \leq 5$, these waves are always stable. However, if $p > 5$ there exists a critical speed c_p^+ such that the positive solitary waves are stable for $c > c_p^+$, and they are unstable for $1 < c < c_p^+$. This result was proved by Souganidis and Strauss [16] using the general theory of Albert, Bona, Grillakis, Souganidis, Shatah and Strauss laid

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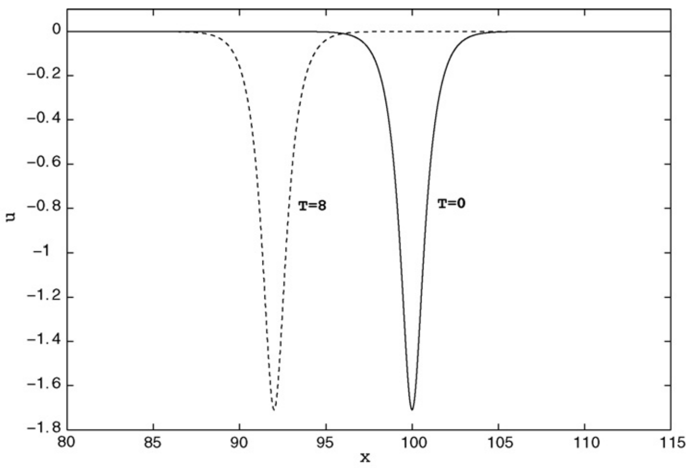

Fig. 1. Solitary wave of depression with speed $c=-1$. Here $p=4$.

down in [9,11], and pioneered by Boussinesq, Benjamin and coworkers [4,7,10,15]. For a thorough review of the results, and a numerical study of the stability of positive solitary waves, the reader may refer [8].

As also observed in [8], if $p\geq 2$ is odd, $-\Phi_c(x-ct)$ is also a solution of the generalized BBM equation. This follows immediately from the fact that $-u$ also satisfies Eq. (1.1). Thus, these are negative waves which also propagate to the right at the positive velocity $c>1$. Consequently, these negative solitary waves are always stable when $p\leq 5$. Otherwise, for $p>5$ these negative solitary waves are stable for speeds $c > c_p^+$, and they are unstable for $1 < c < c_p^+$.

Finally, note that if $p\geq 2$ is even, then the formula (1.3) is valid for $c<0$ as well, resulting in negative solitary waves which propagate to the left. Surprisingly, these solitary waves can be unstable even if $p<5$. The main contribution of this paper is a numerical study of the stability of these negative solitary waves for both subcritical and supercritical p. Note that the case $p=2$ was already treated by one of the authors in [12].

A representative negative solitary wave is shown in Fig. 1, where $p=4$ and $c=-1$. In this calculation, the solution was approximated numerically from $T=0$ to $T=8$ and the size of the domain was 200. While Eq. (1.1) with quadratic and cubic nonlinearity is known to be a reasonable model for long waves of small-amplitude and negligible transverse variation, it is also apparent that the amplitudes of the negative solitary waves are of order 1. Since the equation is given in dimensional variables, these solutions do not fall into the regime of physical validity of the equation as a long-wave model. In fact, this agrees with the observation explained in [5] that solitary waves of depression do not occur on the surface of fluids unless surface tension is very strong.

2. Orbital stability

As already observed by Benjamin and coworkers [4,6], a solitary wave cannot be stable in the strictest sense of the word. To understand this, consider two solitary waves of different height, centered initially at the same point. Since the two waves have different amplitudes they have different velocities according to the expression (1.3). As time passes the two waves will drift apart, no matter how small the initial difference was. However, in the situation just described, it is evident that two solitary waves with slightly differing height will stay similar in shape during the time evolution. Measuring the difference in shape therefore will give an acceptable notion of stability. This sense of orbital stability was introduced by Benjamin [4]. We say a solitary wave is orbitally stable if a solution u of Eq. (1.1) that is initially sufficiently close to a solitary wave will always stay close to a translation of the solitary-wave during the time of evolution. A more mathematically precise definition is as follows. For any $\varepsilon>0$, consider the tube

$$U_\varepsilon = \{u\in H^1 : \inf_s \|u-\tau_s\Phi_c\|_{H^1} < \varepsilon\},$$

where $\tau_s f(x)=f(x+s)$ is a translation of f. The set U_ε is an ε-neighborhood of the collection of all translates of Φ_c.

Definition 2.1. The solitary wave is **stable** if for all $\varepsilon>0$, there exists a $\delta>0$ such that if $u_0=u(\cdot,0)\in U_\delta$, then $u(\cdot, t)\in U_\varepsilon$, for all $t\in\mathbb{R}$. The solitary wave Φ_c is **unstable** if Φ_c is not stable.