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Modified finite difference schemes for geophysical flows

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Abstract

In previous works we developed a method to improve both the accuracy and computational efficiency of a given finite difference scheme used to simulate a geophysical flow. The resulting modified scheme is at least as accurate as the original, has the same time step, and often uses the same spatial stencil. However, in certain parameter regimes it is higher order. In this paper we apply the method to more realistic settings. Specifically, we apply the method to the Navier–Stokes equations and to a sea breeze model.

Keywords: Geophysical flows; Higher order finite difference schemes; PDEs

1. Introduction

Many have argued the use of higher-order finite difference schemes in geophysical flows is a more efficient way to increase the accuracy/dynamical properties than an increase in the spatial grid resolution [3,5,7,10,9,15,16]. Henshaw and coauthors, in [5] and related papers, have shown the advantage of higher-order transport methods for the Navier–Stokes equations for flows over simple topography, identifying fourth-order accuracy as particularly advantageous. Iskandarani et al., [7], have come to similar conclusions for more oceanographically-relevant topographic configurations using a spectral element approach. Of course traditional higher-order schemes require wider spatial stencils which can complicate the approximations near a physical boundary, complicate the implementation of an implicit scheme, and in some cases increase the minimum size time step required for stability of an explicit scheme. These obstacles have led to the construction of *compact difference* and *well-balanced* schemes; In the current context see [1,6,11,13] and the references therein. Compact difference schemes are quite general and apply in many applications. They also require the inversion of tridiagonal matrices. Well-balanced schemes are highly specific to a given problem and specific balance.

The higher-order finite difference schemes proposed here are in some sense a blend of well-balanced and compact schemes. Technically we do not introduce new schemes—only a way of modifying a given scheme to make it higher order in certain parameter regimes. The procedure is particularly effective for geophysical flows where a balance not involving the time derivative occurs. The higher-order scheme is constructed by modifying the truncation error of the original finite difference scheme. Specifically, the form of the governing equations at steady state is used to replace higher-order derivatives in the truncation error with low-order derivatives.

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The procedure for modifying the truncation error has successfully been applied to many finite difference schemes including approximations to the shallow-water equations and transport equations, [8,10,9,15]. In each case the extra cost, in appropriate parameter spaces, required to modify the truncation error is more than compensated by the increase in accuracy. The process leaves stability properties as well as conserved quantities of the original scheme unchanged; In many cases the modified scheme resides on the same spatial stencil, or else it allows flexibility in choosing the terms that reside on larger stencils; Moreover, in some cases the modified scheme is sign preserving for all parameter values, even when the original scheme is not. Finally, while we have no proof the modified scheme is never less accurate than the scheme from which it is derived, our experience indicates that the modifications do no harm to the starting scheme.

The implementation of the method is highly specific to the problem and finite difference scheme to which it is applied. In this paper we advance the method closer to realistic settings by applying the idea to the two-dimensional Navier–Stokes equations and to a sea breeze model. Accommodating the non local properties of the Navier–Stokes equations presents some challenges not encountered in our previous work. Moreover, the generality in which the method is applied should make it possible to extend it to intermediate models (geostrophic approximations), and the three-dimensional Navier–Stokes and related equations.

2. Burgers' equation

To modify a given finite difference scheme approximating a PDE, we first compute the truncation error—the error made by replacing continuous derivatives with discrete differences. As a result of using Taylor's theorem, the truncation error contains derivatives of higher order than present in the original PDE. To make a given scheme more accurate, some of these terms must be eliminated. Since in many geophysical flows, the time derivative is not the dominant term (the fluid is geostrophically balanced, hydrostatically balanced...), we use the PDE at steady state to replace higher-order derivatives with lower-order derivatives. The result yields a higher-order scheme on the same mesh as the original finite difference scheme.

To illustrate the basic concept of the procedure consider Burgers equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \lambda u_{xx} + u u_x = f, \quad 0 < x < L,\ t > 0,\\
&u(0,t) = 0 = u(L,t), \quad t > 0.
\end{aligned}$$

The constant λ is positive. We assume a uniform spatial grid of width Δx, and to shorten the exposition, we employ the notation

$$\delta_{xx} u_i = \frac{1}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1}), \qquad \delta_x u_i = \frac{1}{2\Delta x}(u_{i+1} - u_{i-1}).$$

Suppose we are given the second-order scheme, differenced in conservative flux form

$$\frac{du_i}{dt} - \lambda \delta_{xx} u_i + \frac{u_{i+1}^2 - u_{i-1}^2}{4\Delta x} = f_i, \quad 1 \le i \le N-1,$$

where N is given. The numbers u_i approximate the nodal values of the solution to Burgers equation at $x_i = iL/N = i\Delta x$. Moreover, we leave the time derivative continuous since the specific time integrator used is not important—we are only modifying the spatial truncation error. Temporal errors are assumed to be small.

We want to make the scheme higher order without making the scheme less accurate, affecting the time step for stability, or widening the stencil. To be more consistent with parameters in a geophysical setting, we suppose the viscous term is small compared to the other terms and consequently ignore it in the truncation analysis. Thus the starting scheme leads to

$$\begin{aligned}
0 &= u_t - \lambda u_{xx} + u u_x - f\\
&= u_t - \lambda \delta_{xx} u_i + \frac{1}{4}\delta_x u_i^2 - f + \left(u u_x - \frac{1}{4}\delta_x u_i^2\right)\\
&= u_t - \lambda \delta_{xx} u_i + \frac{1}{4}\delta_x u_i^2 - f - \left(\frac{u u_{xxx}}{6} + \frac{u_x u_{xx}}{2}\right)\Delta x^2 + \mathcal{O}(\Delta x^4). \qquad (1)
\end{aligned}$$