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Multiscale cell-based coarsening for discontinuous problems

Alfonso Limon [a], Hedley Morris [b,*]

[a] School of Mathematical Sciences, Claremont Graduate University, 710 N. College Ave., Claremont, CA 91711, United States
[b] Department of Mathematics, San Jose State University, One Washington Square, San Jose, CA 95192, United States

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Abstract

Whether tracking the eye of a storm, the leading edge of a wildfire, or the front of a chemical reaction, one finds that significant change occurs at the thin edge of an advancing line. The tracking of such change-fronts comes in myriad forms with a wide variety of applications expressible as PDEs. Expanding on Ami Harten's ideas, we construct an alternative to wavelet-based grid refinement, a multiresolution coarsening method that is capable of capturing sharp gradients across different scales, thus improving PDE-based simulations by concentrating computational resources where the solution varies sharply. We present this alternative grid coarsening method and compare its performance to other multiresolution methods by means of several examples.

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1. Introduction

There are many adaptive wavelet-based PDE methods [3,8–10,19,21,25]. Our alternative method, an expansion of Ami Harten's generalized wavelets [18], yields a multiresolution coarsening procedure that captures sharp gradients across different scales and improves PDE-based simulations by concentrating computational nodes where the solution changes abruptly. Our method has also proved useful in flagging points near jumps that would benefit from adaptive stencil selection strategies, such as those proposed by Harten and Osher [20,23]. In the present paper, we use a linear version of Harten's multiresolution analysis [1,30] to construct a multilevel based front tracking scheme. This scheme, dubbed the Multilevel Front Tracking method, or MFT, works via detail coefficient thresholding [15]. These detail coefficients are used to adapt the grid near the jump condition, thereby providing a coarser version that captures the essential features of the original solution. Our MFT grid coarsening scheme thus provides accurate derivative information to solve PDEs with discontinuous solutions. The goal of the present project is to provide an adaptive computational platform based on generalized multiresolution analysis that is able to discretize a PDE, refine the solution and solve the resulting linear system all under one unified framework.

Section 2 introduces Harten's multiresolution analysis and proceeds to a detailed description of our MFT method's discontinuity localization strategy in Section 3. Sections 4 and 5, respectively, cover the localization of jumps after the

* Corresponding author.
E-mail addresses: alfonso.limon@cgu.edu (A. Limon), morris@math.sjsu.edu (H. Morris).

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tiling of the computational domain and the development of the multilevel front tracking method. Finally, in Section 6, our MFT method is compared with other multilevel methods both as a coarsening strategy and later as an adaptive PDE solver. We end with a short summary and discussion of our development road map in Section 7.

2. Multiresolution analysis

Multiresolution schemes are a natural framework by which to locate fast transitions across multiple scales. Wavelets [11,15,24,31] and their Lifted extensions [13,22,32,33] form the basis for many multiresolution schemes. In addition to these, there are nonlinear counterparts to Lifting [6,7,16,17,26] and adaptive stencil selection methods, such as those first proposed by Harten [18] and later extended by Aràndiga and Donat [1] and Schröder-Pander et. al. [30]. We use these extensions to construct a cell-based coarsening method, and what follows is a short introduction to Harten's multiresolution analysis.

Suppose that we have a space of functions $\mathcal{F} \subset \{f | f : \Omega \subset \mathbb{R}^m \to \mathbb{R}\}$, where Ω is a bounded region, and that $\forall f \in \mathcal{F}$ there is a discretization operator $\mathcal{D}^k : \mathcal{F} \to V^k$, where V^k is a finite linear space with dimension k. The objective is to design a multiresolution scheme specifically adapted to sequences obtained from $\mathcal{D}^k$. This is achieved through the introduction of reconstruction operators $\mathcal{R}^k : V^k \to \mathcal{F}$. The operators, $\mathcal{D}$ and $\mathcal{R}$, can be constructed to provide different resolution details

$$V^k \underset{\mathcal{R}^k}{\overset{\mathcal{D}^k}{\leftrightarrows}} \mathcal{F} \underset{\mathcal{D}^{k-1}}{\overset{\mathcal{R}^{k-1}}{\leftrightarrows}} V^{k-1}, \tag{2.1}$$

where $dim(V^{k-1}) < dim(V^k)$.

Given V^k and V^{k-1}, a decimation operator is defined $D_k^{k-1} : V^k \to V^{k-1}$, which lowers the resolution level from k to $k-1$. Inverting the process, a prediction operator $P_{k-1}^k : V^{k-1} \to V^k$ increases the resolution from $k-1$ to k. Therefore, the multiresolution framework

$$V^k \underset{D_k^{k-1}}{\overset{P_{k-1}^k}{\leftrightarrows}} V^{k-1} \tag{2.2}$$

acts on a sequence $v^k \in V^k$ constructed by a discretization process $v^k = \mathcal{D}^k f$ at resolution level k, and by definition, $D_k^{k-1}(\mathcal{D}^k f) = \mathcal{D}_{k-1} f$, so $D_k^{k-1} = \mathcal{D}_{k-1}\mathcal{R}^k$. However, the sequence is nested, $\mathcal{D}^k f = 0 \Rightarrow \mathcal{D}^{k-1} f = 0, \forall f \in \mathcal{F}$, so D_k^{k-1} cannot depend on the reconstruction operator $\mathcal{R}^k$; thus, D_k^{k-1} is linear. Furthermore, the prediction operator P_{k-1}^k is the right-inverse of D_k^{k-1} and so $P_{k-1}^k = \mathcal{D}^k\mathcal{R}^{k-1}$.

Using this framework, $v^k \in V^k$ can be approximated via the information content at level $k-1$, i.e., $P_{k-1}^k D_k^{k-1} : V^k \to V^k$, and errors in the approximation can be computed by $e^k = (I_{V^k} - P_{k-1}^k D_k^{k-1})v^k$. This constructs a one-to-one correspondence between v^k and $[e^k, v^{k-1}]$, but e^k is in the null space of D_k^{k-1}, because $D_k^{k-1}e^k = D_k^{k-1}v_k - (D_k^{k-1}P_{k-1}^k)v_{k-1} = 0$. As a consequence, expressing e_k in terms of a basis in V^k results in redundant information. However, this redundant information can be discarded by projecting onto $\mathcal{N}(D_k^{k-1})$ and expressing the prediction error as $e^k = \sum_j d_j^k \mu_j^k \equiv E_k d^k$, where μ_j^k spans $\mathcal{N}(D_k^{k-1})$. Defining the assignment operator G^k as $E^k G^k = I_{\mathcal{N}(D_k^{k-1})}$ results in a less redundant correspondence between v^k and $[d^k, v^{k-1}]$, where the detail coefficients $d^k = G^k e^k$ are the projection errors expressed by any basis of $\mathcal{N}(D_k^{k-1})$. In other words, d^k represents the information at level k that cannot be predicted by P_{k-1}^k at level $k-1$. The pyramid scheme,

$$\begin{array}{ccccccc} v^k & \to & v^{k-1} & \to & v^{k-2} & \to & \cdots \\ & \searrow & & \searrow & & \searrow & \\ & & d^k & \to & d^{k-1} & \to & \cdots \end{array} \tag{2.3}$$

expands this two-level scheme to multiple resolution levels; refer to [1,30] for further details. In Section 4, we develop a coarsening strategy for mesh refinement based on d^k as a variational measure of f at scale k, but first, the jump-points must be localized.