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Statistical moments of the solution of the random Burgers–Riemann problem

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Abstract

We solve Burgers' equation with random Riemann initial conditions. The closed solution allows simple expressions for its statistical moments. Using these ideas we design an efficient algorithm to calculate the statistical moments of the solution. Our methodology is an alternative to the Monte Carlo method. The present approach does not demand a random numbers generator as does the Monte Carlo method. Computational tests are added to validate our approach.
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1. Introduction

When the data of a differential equation, the coefficients or the initial conditions, are random variables its solution is a random function; this kind of mathematical problem has been called a random differential equation. A great number of practical processes under current investigations falls on the stochastic modeling; we may quote the models in control, communications, economic systems, chemical kinetics, biosciences, statistical mechanics and spatial areas and so on. The methodology to understand and solve differential equations with uncertainties has stimulated studies under several points of view. Since the solution is a random function, one particular solution corresponding to a specific realization is not of concern: it is important to know the statistical properties of the solution such as its mean, variance, or other statistical moments.

Some methods for random differential equations are categorized as moment equations methods. In these methods the purpose is to obtain differential equations governing the statistical moments. The most important of these equations is the differential equation for the expectation (mean), which is called for some authors as effective equation. As far as we know no effective equation is known for the nonlinear problem discussed in this paper.

The Monte Carlo method is an alternative in solving random differential equations. Partial differential equations and the Monte Carlo method have been related for more than a century, since the works developed by Rayleigh [18], Courant et al. [2], and Kolmogorov [12]. For instance, Courant et al. showed that a particular finite difference equation for the two-dimensional Dirichlet boundary value problem and a two-dimensional random walk produce the same

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results. In modern terms the Monte Carlo method originated from Los Alamos and the atomic bomb project. Now it is being used in many scientific fields [6,20]. The basic idea is to solve a large number of deterministic differential equations choosing particular values for the random variables according to their assumed probabilistic distribution. The statistical information of the random solution is estimated using these realizations. The Monte Carlo method can be used in either linear or nonlinear random differential equations. However, the exceptionally large volume of calculations, and the difficulty for generating random numbers limit the significance of this method.

In a different direction we have been studying numerical methods for the random transport equation. In the linear case our ideas were inspired by Godunov's method [9,15] for the deterministic transport equation. In [3], we present an explicit expression for the random solution to one-dimensional random advective equations where the constant velocity and the Riemann initial condition are random variables. This closed solution yields simple expressions for its statistical moments, and computational experiments show good agreement between our expressions and the Monte Carlo method for the first three moments. The closed solution for random Riemann problems and Godunov's ideas are used in [4,5] to design numerical methods to calculate the mean and variance of the solution to transport equations with more general initial condition (random fields). Our methods are explicit and do not demand differential equations governing the statistical moments, the effective equations. Furthermore, our scheme is consistent and stable with the diffusive effective equation presented in the literature [8]. Computational experiments have shown good agreements with the Monte Carlo method.

In this paper, we generalize our previous ideas to solve the random Riemann problem for Burgers' equation

$$\frac{\partial}{\partial t}U(x,t)+\frac{1}{2}\frac{\partial}{\partial x}U^2(x,t)=0,\quad t>0,\ \ x\in\mathbb{R},$$

$$U(x,0)=\begin{cases}U_{\mathrm{L}}, & \text{if}\quad x<0,\\ U_{\mathrm{R}}, & \text{if}\quad x>0,\end{cases}\tag{1}$$

where U_{L} and U_{R} are random variables. Here the randomness appears only because of the initial condition. The deterministic version of (1) was introduced by Burgers [1] as the simplest model that captures some key features of gas dynamics, the nonlinear hyperbolic term. But, rather than modeling a physical process, the inviscid Burgers equation has been widely used for developing both theoretical and numerical methods in the literature of deterministic hyperbolic equations.

Taking into account that several numerical methods to deal with deterministic conservation laws use solutions of Riemann problems (*Random Choice Method* developed by Glimm [7], and *Godunov's method* [9,15], for example), we believe that the results of the current paper may be useful in developing numerical methods for more general random conservation laws. Moreover, since the mathematical theory of methods to random partial differential equations are difficult and not complete yet (see [13,16,19], for example), numerical methods can be a good alternative to deal with random differential equations.

Kim [11] presents a scheme to calculate the statistical moments of the random Burgers' equation. Nevertheless, the author considers the simple case where the random initial condition is an explicit function of the spatial variable and of the normal random variable with zero mean and unit variance. The author uses Wiener chaos expansion to separate random and deterministic effects, and utilizes the Lax-Wendroff method to discretize the deterministic system of partial differential equations that governs the propagation of randomness.

In this paper, we use two basic ideas to construct the solution, and its moments, to (1): (i) the realizations of the probabilistic problem are nonlinear transport equations whose analytical solutions are known (shock and rarefaction waves); and (ii) the random solution and its statistical moments, as functions of the initial condition and its joint density function, are found using geometrical partitions of the phase plane (U_{L}, U_{R}). Integrations on these sets are the shock and rarefaction averaging process.

The outline of this paper is as follows. In Section 2, we deduce an explicit solution to problem (1). We also show the similarity of the solution as well as present an expression for its statistical moments. Based on bidimensional midpoint quadrature formula, in Section 3 we suggest an efficient algorithm to approximate the statistical moments. Finally, we present some computational tests and conclusions.

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