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# Multiple stochastic integrals with Mathematica

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**Abstract**

In the construction of numerical methods for solving stochastic differential equations it becomes necessary to calculate the expectation of products of multiple stochastic integrals. Well-known recursive relationships between these multiple integrals make it possible to express any product of them as a linear combination of integrals of the same type. This article describes how, exploiting the symbolic character of Mathematica, main recursive properties and rules of Itô and Stratonovich multiple integrals can be implemented. From here, a routine that calculates the expectation of any polynomial in multiple stochastic integrals is obtained. In addition, some new relations between integrals, found with the aid of the program, are shown and proved.
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## 1. Introduction

Stochastic Taylor expansions, due to Platen and Wagner, see, e.g. [10], express any functional of the solution of a stochastic differential equation as a sum that involves multiple stochastic integrals. Stochastic Taylor expansions have been widely used in the construction of numerical methods for stochastic differential equations. Since the integrals that appear in a Itô or Stratonovich Taylor expansion contain the information of the Wiener process, a numerical scheme must include these integrals or random variables representing them to achieve the desired order of convergence, see, e.g. [1,7] or [9] and their references. It is then necessary to be able to determine the expected value of multiple stochastic integrals as well as of their products. Some of these expectations can be directly determined. For example, for Itô integrals Kloeden and Platen [7] proved that the expectation of a product is 0 if the total number of one-dimensional Wiener processes involved in the integrals (each one counted with its multiplicity) is odd. Burrage [1] proved the same property for Stratonovich integrals. In addition, the expectation of any product of two Itô integrals is known. For a general product the approach is to express it as a sum of simpler terms whose expectations can be calculated. Well-known recursive relationships between multiple integrals makes it possible to express any product of them as a linear combination of integrals of the same type. This kind of job is specially amenable to the use of symbolic manipulation software. Mathematica, see [11], is a powerful computer package, specially suited to the algebraic computations that appear in stochastic calculus. Some parts of stochastic Itô calculus have been implemented within Mathematica by Kendall, see [4]. Recently the same author has extended the initial package *Itovsn3* to compute the expectations of stochastic integrals based on a one-dimensional Wiener process, see [5]. Other general-purpose mathematics software

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packages like MAPLE have been used for stochastic calculus and, specifically, in relation to multiple stochastic integrals, see, e.g. [1,2] or [8].

This article describes how, exploiting the symbolic character of Mathematica, main recursive properties and rules of Itô and Stratonovich multiple integrals can be implemented. From here, a short routine that calculates the expectation of any polynomial in multiple Itô or Stratonovich integrals is obtained. In addition, some new relations between integrals, found with the aid of the program, are shown and proved. The job is partly inspired by the work of Burrage [1], where a long set of routines in MAPLE are presented to do, with some limitations, the tasks here described.

## 2. Multiple stochastic integrals

Let $(\Omega, \mathcal{A}, P)$ denote a complete probability space with $\mathcal{A} = \{\mathcal{A}_t, t \geq 0\}$ an increasing right continuous family of complete sub-$\sigma$-algebras of $\mathcal{A}$. Let $W = (W_t^1, \ldots, W_t^m)$ be a standard Wiener process. As usual, Itô and Stratonovich integration with respect to the component $W_t^j$ will be denoted by $\mathrm{d}W_t^j$ and $\circ\mathrm{d}W_t^j$, respectively. It is also usual to write $\mathrm{d}W_t^0$ instead $\mathrm{d}t$ for Lebesgue integrals.

Given $l \in \mathbf{N}$, a vector of the form $\alpha = (j_1, \ldots, j_l)$ with $j_1, \ldots, j_l \in \{0,1, \ldots, m\}$ is called a multi-index of length $l(\alpha) = l$. For $\alpha = (j_1, \ldots, j_l)$, $\alpha$—denotes the multi-index $(j_1, j_2, \ldots, j_{l-1})$, with the understanding that $(j_1)- = v$ is a multi-index of length equal to 0. If $\alpha = (j_1, \ldots, j_l)$, it will be denoted by $I_{\alpha,t}$, or just $I_\alpha$ when $t$ is obvious, the multiple Itô integral:

$$I_\alpha = \int_{t_0}^{t} \left( \int_{t_0}^{s_l} \cdots \int_{t_0}^{s_3} \left( \int_{t_0}^{s_2} 1 \, \mathrm{d}W_{s_1}^{j_1} \right) \mathrm{d}W_{s_2}^{j_2} \cdots \right) \mathrm{d}W_{s_l}^{j_l}.$$

The first task of our program is to express any product of multiple integrals as a sum of integrals of the same type. To take advantage of the symbolic facilities of Mathematica, the multi-indices $\alpha = (j_1, \ldots, j_l)$ will be treated as lists and the multiple integral $I_\alpha$ will be written as a function $i[\alpha]$. So, we will write `i[{1, 0, 1}]` to represent $I_{(1,0,1)}$.

There exists two results, see the following lemmas, that determine important relations between multiple Itô integrals.

**Lemma 1** ((Kloeden–Platen [6])). *If* $j, j_1, \ldots, j_l \in \{0,1, \ldots, m\}$ *then*

$$I_{(j)} I_{(j_1,\ldots,j_l)} = \sum_{i=0}^{l} I_{(j_1,\ldots,j_i,j,j_{i+1},\ldots,j_l)} + \sum_{i=1}^{l} \chi_{\{j_i = j \neq 0\}} I_{(j_1,\ldots,j_{i-1},0,j_{i+1},\ldots,j_l)},$$

*where* $\chi_A$ *denotes the indicator function of the set* $A$.

Then our first routine applies this lemma to calculate the product of two Itô integrals $I_\alpha * I_\beta$, one of them simple.

```
In[1]:= Delta[j_, k_] := Sign[j]*KroneckerDelta[j, k];
        Unprotect[Times];
        i[x_]*i[alpha_List] :=
            Sum[i[Insert[alpha, x, k]], k, 1, Length[alpha] + 1] +
              Sum[Delta[x, alpha[[k]]]*i[ ReplacePart[alpha, 0, k]],
                  k, 1,Length[alpha]];
```

Now we can calculate $I_{(1)} I_{(0,1,1)}$ as a sum of multiple integrals:

```
In[4]:= i[{1}]*i[{0,1,1}]
Out[4]:= i[{0,0,1}]+i[{0,1,0}]+3i[{0,1,1,1}]+i[{1,0,1,1}]
```

which means $I_{(1)} I_{(0,1,1)} = I_{(0,0,1)} + I_{(0,1,0)} + 3I_{(0,1,1,1)} + I_{(1,0,1,1)}$.

Notice that for simplicity we have used the symbol (*), i.e. the built-in function `Times` of Mathematica, to define the product of two integrals. Then we have had to unprotect it; and the new product has the attributes of the ordinary product `Times`. In particular, it is commutative and associative. But it does not effectively implement the distributive law, e.g. it does not calculate explicitly $I_{(1)}(I_{(0)} + 2I_{(1,1)})$. Notice also that by default the powers of integrals, e.g. $I_{(1)}^3$, are not calculated explicitly as a iterative product. Then we have to force these tasks adding the lines:

```
In[5]:= x_*(y_ + z_) := x*y + x*z;
        Unprotect[Power];
        i[x_List]^n_Integer := Fold[Times,i[x],Table[i[x],k,n - 1]]/; n > 0;
        (x_ + y_)^n_Integer := Fold[Times,x + y,Table[x + y,k,n - 1]]/; n > 0;
        Protect[Power];
```

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