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High-order compact boundary value method for the solution of unsteady convection–diffusion problems

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Abstract

In this paper, we propose a new class of high-order accurate methods for solving the two-dimensional unsteady convection–diffusion equation. These techniques are based on the method of lines approach. We apply a compact finite difference approximation of fourth order for discretizing spatial derivatives and a boundary value method of fourth order for the time integration of the resulted linear system of ordinary differential equations. The proposed method has fourth-order accuracy in both space and time variables. Also this method is unconditionally stable due to the favorable stability property of boundary value methods. Numerical results obtained from solving several problems include problems encounter in many transport phenomena, problems with Gaussian pulse initial condition and problems with sharp discontinuity near the boundary, show that the compact finite difference approximation of fourth order and a boundary value method of fourth order give an efficient algorithm for solving such problems.

Keywords: Unsteady convection–diffusion equation; Compact finite difference scheme; Boundary value methods; High accuracy; Method of lines

1. Introduction

Consider the two-dimensional unsteady convection–diffusion equation

$$\frac{\partial u}{\partial t}(x, y, t) + \beta_x \frac{\partial u}{\partial x}(x, y, t) + \beta_y \frac{\partial u}{\partial y}(x, y, t) = \alpha_x \frac{\partial^2 u}{\partial x^2}(x, y, t) + \alpha_y \frac{\partial^2 u}{\partial y^2}(x, y, t),$$
$$(x, y, t) \in [0, L] \times [0, L] \times [0, T], \tag{1.1}$$

with initial condition

$$u(x, y, 0) = \phi(x, y),$$

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and the boundary conditions

$$\begin{aligned} &u(x,0,t)=f_0(x,t), \qquad u(x,L,t)=f_1(x,t), \qquad t\geq 0,\\ &u(0,y,t)=g_0(y,t), \qquad u(L,y,t)=g_1(y,t), \qquad t\geq 0, \end{aligned} \tag{1.2}$$

where $u(x, y, t)$ is a transported (advected and diffused) scalar variable, β_x and β_y are arbitrary constants which show the speed of convection and the diffusion coefficients, i.e. α_x and α_y are positive constants.

For a positive integer n let $h = L/n$ denote the step size of spatial derivatives and Δt for step size of temporal derivative. So we define

$$\begin{aligned} &x_i = ih,\ y_j = jh, \quad i, j = 0, 1, 2, \ldots, n,\\ &t_k = k\Delta t, \qquad\qquad k = 0, 1, 2, \ldots, N. \end{aligned}$$

This equation may be seen in computational hydraulics and fluid dynamics to model convection–diffusion of quantities such as mass, heat, energy, vorticity, etc. The applications of Eq. (1.1) have been stated in [7,8,18].

Various finite difference schemes have been proposed to solve convection–diffusion problems approximately. For solving steady convection–diffusion equations, some compact finite difference approximations have been developed in the last two decay [12,20,16]. The authors of [17] proposed a nine-point high-order compact implicit scheme for Eq. (1.1) which is third-order accurate in space and second-order accurate in time, and has a large zone of stability. Spotz and Carey [21] presented an extension of higher order compact difference techniques for steady-state [20] to the time-dependent problems. These methods are conditionally stable and have fourth-order accuracy in space and second order or lower in time component. Cecchi and Pirozzi [5] presented a family of unconditionally stable finite difference methods for Eq. (1.1) and the possible maximum order of accuracy is $O(h^3, \Delta t^2)$. Other new and high-order formulas for solving Eq. (1.1) have been developed by some authors [14,15,23]. Kalita et al. [14] proposed some schemes for solving Eq. (1.1) which are based on high-order compact scheme and weighted time discretization. Their schemes are second or lower order accurate in time and fourth-order accurate in space. Karaa and Zhang [15] presented a high-order alternating direction implicit (ADI) method for solving Eq. (1.1) which is fourth and second order in space and time respectively and is unconditionally stable. More research works can be found in Refs. [10,23,25–27]. In this paper, we introduce an unconditionally stable scheme for Eq. (1.1) which is fourth-order accurate in both space and time variables. We apply a compact finite difference scheme of fourth order for discretizing spatial derivatives and boundary value methods for the time integration of Eq. (1.1). Boundary value methods (BVMs) are unconditionally stable and are high-accuracy schemes for solving ordinary differential equations based on the linear multi-step formulas [2–4]. Unlike Runge–Kutta or other initial value methods (IVM), BVMs achieve the advantage of both good stability and high-order accuracy [1–4,9,11].

The outline of this paper is as follows: in Section 2, we present the fourth-order compact finite difference approximation of one and two-dimensional steady-state convection–diffusion equations. In Section 3 we briefly introduce the boundary value methods and present a fourth-order boundary value method for solving IVPs. In Section 4, we apply the fourth-order compact difference scheme for discretizing spatial derivatives and the fourth-order boundary value method for the resulted linear system of ordinary differential equations. Also in this section we propose a scheme for the solution of resulted large linear system of equations. In Section 5 we describe the numerical experiments of solving one and two-dimensional unsteady convection–diffusion equations with the method of this article for four problems and compare the numerical results with some well-known numerical methods. The concluding remarks in drawn in Section 6. Some references are given for possible extension of the approach [28–30].

2. Compact finite difference schemes

The basic approach for high-order compact difference methods is to introduce the standard compact difference approximations to the differential equations and then by repeated differentiation and associated compact differencing, a new high-order compact scheme will be developed that incorporates the effect of the leading truncation error terms in the standard method [20]. Recently due to the high order, compactness and high resolution, we have seen increasing population for high-order compact difference methods in computational fluid dynamics, computational acoustics and electromagnetic [19,20]. In this section we state the fourth-order compact finite difference scheme for the spatial derivatives of (1.1).