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Full convergence of an approximate projection method for nonsmooth variational inequalities

J.Y. Bello Cruz [a,*], A.N. Iusem [b]

[a] Instituto de Matemtica e Estatstica, Universidade Federal de Goiás, Campus Samambaia, CEP 74001-970 GO, Goiânia, Brazil
[b] Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, Jardim Botânico, CEP 22460-320 RJ, Rio de Janeiro, Brazil

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Abstract

We analyze an explicit method for solving nonsmooth variational inequality problems, establishing convergence of the whole sequence, under paramonotonicity of the operator. Previous results on similar methods required much more demanding assumptions, like coerciveness of the operator.

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1. Introduction

Let C be a nonempty, closed and convex subset of $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ a point-to-set operator. The variational inequality problem for T and C, denoted VIP(T, C), is the following:

Find $x^* \in C$ such that there exists $u^* \in T(x^*)$ satisfying

$$\langle u^*, x - x^* \rangle \geq 0 \quad \forall\, x \in C.$$

We denote the solution set of this problem by $S(T, C)$.

The variational inequality problem was first introduced by Hartman and Stampacchia [13] in 1966. An excellent survey of methods for finite dimensional variational inequality problems can be found in [10].

Here, we are interested in direct methods for solving VIP(T, C). They are called direct because the solution of subproblems at each iteration is not required. Iterate x^{k+1} is computed using only information on the previous point x^k and easy computations. The basic idea consists of extending the projected gradient method for constrained optimization, i.e. for the problem of minimizing $f(x)$ subject to $x \in C$. This problem is a particular case of VIP(T, C) taking $T = \nabla f$. This procedure is given by the following iterative scheme:

$$x^0 \in C, \tag{1}$$

* Corresponding author.
E-mail addresses: yunier@impa.br (J.Y. Bello Cruz), iusp@impa.br (A.N. Iusem).

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$$x^{k+1} = P_C(x^k - \alpha_k \nabla f(x^k)), \tag{2}$$

with $\alpha_k > 0$ for all k. The coefficients α_k are called stepsizes and $P_C : \mathbb{R}^n \to C$ is the orthogonal projection onto C, i.e. $P_C(x) = argmin_{y \in C} ||x - y||$.

An immediate extension of the method (1) and (2) to VIP(T, C) for the case in which T is point-to-set, is the iterative procedure given by

$$x^0 \in C, \tag{3}$$

$$x^{k+1} = P_C(x^k - \alpha_k u^k), \tag{4}$$

where $u^k \in T(x^k)$, and the sequence $\{\alpha_k\}$ of step-sizes is adequately chosen.

Convergence results for this method require some monotonicity properties of T. We introduce next several possible options.

Definition 1. Consider $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ and $W \subset \mathbb{R}^n$ convex. T is said to be:

(i) monotone on W if $\langle u - v, x - y \rangle \geq 0$ for all $x, y \in W$ and all $u \in T(x)$, $v \in T(y)$,
(ii) paramonotone on W if it is monotone in W, and whenever $\langle u - v, x - y \rangle = 0$ with $x, y \in W$, $u \in T(x)$, $v \in T(y)$ it holds that $u \in T(y)$ and $v \in T(x)$,
(iii) strictly monotone on W if $\langle u - v, x - y \rangle > 0$ for all $x, y \in W$ such that $x \neq y$, and all $u \in T(x)$, $v \in T(y)$,
(iv) uniformly monotone on W if $\langle u - v, x - y \rangle \geq \psi(||x - y||)$ for all $x, y \in W$ and all $u \in T(x)$, $v \in T(y)$, where $\psi : \mathbb{R}_+ \to \mathbb{R}$ is an increasing function, with $\psi(0) = 0$,
(v) strongly monotone on W if $\langle u - v, x - y \rangle \geq \omega ||x - y||^2$ for some $\omega > 0$ and for all $x, y \in W$ and all $u \in T(x)$, $v \in T(y)$.

It follows from Definition 1 that the following implications hold: (v) ⇒ (iv) ⇒ (iii) ⇒ (ii) ⇒ (i). The reverse assertions are not true in general.

Convergence of the scheme (3)–(4), is established in [1] assuming uniform monotonicity of T, and in [3] assuming paramonotonicity of T.

We remark that there is no chance to relax the assumption on T to plain monotonicity, while preserving the convergence properties of the scheme given by (4). For example, consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x) = Ax$, with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. T is monotone and the unique solution of VIP(T, C) is $x^* = 0$. However, it is easy to check that $||x^k - \alpha_k T(x^k)|| > ||x^k||$ for all $x^k \neq 0$ and all $\alpha_k > 0$, and therefore the sequence generated by (4) moves away from the solution, independently of the choice of the stepsize α_k.

Thus, the scheme (3)–(4) fails to converges for arbitrary monotone operators. In such a case, an available option is Korpelevich's method and its variants, which perform a double-step iteration of the form:

$$y^k = P_C(x^k - \alpha_k T(x^k)) \tag{5}$$

$$x^{k+1} = P_C(x^k - \gamma_k T(y^k)), \tag{6}$$

where y^k is an auxiliary point and the positive sequences $\{\alpha_k\}$ and $\{\gamma_k\}$ satisfy some conditions. See, e.g. [17,19–21]. Other algorithms for VIP(T, C), less directly related to Korpelevich's method, can be found in [4,14,23,24].

In this paper we will deal with one-step iterations, and thus we keep the paramonotonicity assumption. We comment next on this assumption.

The notion of paramonotonicity, which is in-between monotonicity and strict monotonicity, was introduced in [6], and many of its properties were established in [9,16]. Among them, we mention the following:

(i) If T is the subdifferential of a convex function, then T is paramonotone; see Proposition 2.2 in [16].
(ii) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is monotone and differentiable, and $J_T(x)$ denotes the Jacobian matrix of T at x, then T is paramonotone if and only if $\text{Rank}(J_T(x) + J_T(x)^t) = \text{Rank}(J_T(x))$ for all x; see Proposition 4.2 in [16].