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Original article

A two-level stabilized nonconforming finite element method for the stationary Navier–Stokes equations

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Abstract

In this paper, we study an approximation scheme that combines a stabilized nonconforming finite element method and a two-level method to solve the stationary Navier–Stokes equations. Error estimates of optimal order are obtained. Numerical results to check these estimates are presented.

MSC: 35Q10; 65N30; 76D05

Keywords: Navier–Stokes equations; Two-level; Nonconforming finite element method; Error estimate; Numerical results

1. Introduction

It is well known that when the finite element methods are used to solve the Navier–Stokes equations, the approximations for velocity and pressure must satisfy the *inf–sup* condition for them to be stable. Although the low-order mixed finite elements, such as $P_1 - P_1, P_1 - P_0, Q_1 - Q_1, Q_1 - P_0$, do not satisfy this condition, they have the practical importance in scientific computation for their computational simplicity and convenience. Moreover, compared with the conforming finite element methods, the nonconforming finite elements [3,4] are more attractive for discretizing partial differential problems since they posses more favorable stability properties and less support sets. Much work has been devoted to the nonconforming low-order finite element methods. For example, the nonconforming elements proposed by Douglas et al. [5] for the velocity and a piecewise constant element for the pressure were used for the stationary Stokes and Navier–Stokes equations in [2], the same nonconforming elements augmented by the conforming bubbles for the velocity and the discontinuous piecewise linear functions for the pressure were used for the Stokes problem in [17], the nonconforming and conforming piecewise linear polynomial approximations for the velocity and pressure were used for the Stokes equations in [21], and the same finite element approximations were used for the Navier–Stokes equations in [27].

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In practice, the numerical solution of a nonlinear system of algebraic equations arising in the discretization of the Navier–Stokes equations can be very time consuming. Two-level methods aim to compute a discrete approximation of the solution of a nonlinear partial differential equation with less computational work and to preserve the optimal order of convergence; see [25,26] for steady state semi-linear elliptic equations and [6,9,10,18–20] for the steady Navier–Stokes equations. As for the nonstationary Navier–Stokes equations, the two-level finite element semi-discretization scheme has been studied in [6], the full discretization of the two-level finite element method in the space variable and the one-level backward Euler scheme in the time variable have been discussed in [23], and the full discretization of the two-level finite element method in both the space and time variables has been studied in [7,8,14]. Recently, some multi-level strategies have been studied for the nonstationary Navier–Stokes equations in [11–13].

The method we study in this paper is to combine a stabilized nonconforming finite element method and the two-level discretization for solving the two-dimensional steady state Navier–Stokes problem under the assumption of the uniqueness condition. This stabilized finite element method is based on a local Gauss integration method [1,22].

The outline of the paper is as follows: In the next section we introduce a mathematical setting for the stationary Navier–Stokes equations. In Section 3 we recall the conclusions of the stabilized nonconforming finite element method for the stationary Navier–Stokes equations. The optimal error estimates of the two level stabilized nonconforming finite element method are obtained in Section 4. Some numerical results are presented in Section 5, which agree with the theoretical results.

2. A functional setting of the Navier–Stokes equations

Let Ω be a bounded domain in $\Re^2$, assumed to have a Lipschitz continuous boundary $\partial\Omega$. We consider the Navier–Stokes problem. Find the velocity $u=(u_1, u_2)$ and the pressure p defined on Ω such that

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in} \quad \Omega, \\ \operatorname{div} u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where f is the body forces per unit mass and $\nu>0$ is the dynamic viscosity. For a mathematical formulation of this problem, we introduce the following Hilbert spaces

$$X = (H_0^1(\Omega))^2, \qquad Y = (L^2(\Omega))^2, \qquad M = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}.$$

The spaces $(L^2(\Omega))^m$, $m=1, 2, 4$, are endowed with the L^2-scalar product and L^2-norm denoted by $(\cdot\,,\cdot)$ and $\|\cdot\|_0$, respectively. The spaces $H_0^1(\Omega)$ and X are equipped with their usual scalar product $(\nabla u, \nabla v)$ and norm $\|u\|_1$.

The continuous bilinear forms $a(\cdot\,,\cdot)$ and $d(\cdot\,,\cdot)$ are defined by

$$a(u, v) = \nu(\nabla u, \nabla v), \quad \forall u, v \in X; \quad d(v, p) = (\operatorname{div} v, p), \quad \forall v \in X, p \in M,$$

and a generalized bilinear form on $(X, M)\times(X, M)$ by

$$\mathcal{B}((u, p), (v, q)) = a(u, v) - d(u, q) - d(v, p), \quad \forall (u, p), (v, q) \in X \times M.$$

A trilinear form on $X\times X\times X$ is given by

$$b(u; v, w) = ((u\cdot\nabla)v, w) + \frac{1}{2}((\operatorname{div} u)v, w) = \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v), \ \forall u, v, w \in X.$$

It is well known that $b(\cdot\,;\cdot\,,\cdot)$ satisfies the following properties [19,15]:

$$b(u; v, w) = -b(u; w, v), \quad \forall\, u, v, w \in X, \tag{2.2}$$

$$|b(u; v, w)| \le N\|u\|_1\|v\|_1\|w\|_1, \quad \forall\, u, v, w \in X, \tag{2.3}$$

where N is a positive constant depending only on the domain Ω.

Then the continuous Galerkin formulation is to find $(u, p)\in X\times M$ such that

$$a(u, v) - d(v, p) - d(u, q) + b(u; u, v) = (f, v), \quad \forall\, (v, q) \in X \times M.$$