Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 80 (2010) 2168–2176
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

# Nodal finite element approximations for the neutron transport equation

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Available online 7 May 2010

**Abstract**

Two classes of nodal methods, weakly and strongly discontinuous, are introduced and applied to the numerical solution of the neutron transport equation in two-dimensional Cartesian geometry and discrete ordinates. These methods are then applied for the approximation of the solution of a reference problem well known in the nuclear engineering literature.


*Keywords:* Transport equation; $S_N$ approximation; Nodal methods; Finite elements

## 1. Introduction

The neutron transport equation in its discrete-ordinates $S_N$ approximation in two-dimensional Cartesian geometry reads:

$$L\psi_k \equiv \mu_k \frac{\partial \psi_k}{\partial x} + \nu_k \frac{\partial \psi_k}{\partial y} + \sigma_t \psi_k = \sigma_s \sum_{l=1}^{M} \omega_l \psi_l + S_k \equiv Q_k, \quad k = 1, \ldots, M, \tag{1.1}$$

where the unknown is $\psi_k$, the angular neutron flux corresponding to the $k$ th ray of the $S_N$ approximation, $M$ being the total number of rays considered which is given in this case by $N(N+2)/2$. The domain to be considered is of the union of rectangles type and boundary conditions must also be imposed.

Classically, with nodal methods, the domain of interest is decomposed in relatively large homogeneous regions or "nodes", over which each angular flux $\psi$ is approximated by a generalized interpolant with interpolation parameters which are cell and/or edge Legendre moments. This unique interpolant is piecewise continuous using in most cases polynomial shape functions. For a ray in the first quadrant, the possible left and bottom edge parameters are known from the boundary conditions or from the neighboring left and bottom cells. The unknowns are thus the right and top edge parameters as well as the cell ones.

In this paper, two classes of polynomial nodal methods are presented. In essence, both classes of methods lead to discontinuous approximations as they at most conserve some edge moments between adjacent nodes. In the first class of methods which is called *weakly discontinuous* one or several moments of the angular flux are conserved between a

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[1] COFAA, research projects 2006-0967 and 2007-1197 supported by SIP-IPN's.

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doi:10.1016/j.matcom.2010.04.010

given cell and its upstream neighbors. The second class of methods, called *strongly discontinuous*, is fully discontinuous and only has outgoing (at top and right) edge moments as parameters, in addition to possible cell moments.

Before dealing with these methods in detail, some notation and the basic formalism are presented in the next section. The two classes of methods are then developed in two sections and a third one proposes some numerical results, before presenting some conclusions in a last section.

## 2. Notation and basic formalism

Assuming that the domain $\Omega$ of the union of rectangles type has been discretized in $N_e$ nodes or rather *cells* or *elements*, i.e. $\Omega \equiv \Omega_h = \cup_{e=1}^{N_e} \Omega_e$, each cell $\Omega_e$ is mapped onto a reference cell $\hat{\Omega} \equiv [-1, +1] \times [-1, +1]$, as it is traditional with finite element methods. A particular finite element is then defined by a set of degrees of freedom $D$ and a space of functions $S$ with $card(D) = dim(S)$. With degrees of freedom which are cell and/or edge moments as in this paper, we shall speak of *nodal finite elements*. For practical purposes, these moments will be taken as Legendre moments.

To describe $D$ and $S$ in a compact way in the *nodal* case, some notation will be helpful. Let $P_i$ be the normalized Legendre polynomial of degree $i$ over $[-1, +1]$ which satisfies

$$P_i(+1) = 1, \quad P_i(-1) = (-1)^i, \quad \text{and} \quad \int_{-1}^{+1} P_i(x)P_j(x)dx = N_i\delta_{ij}, \tag{2.1}$$

with $N_i = 2/(2i + 1)$. Define moreover $P_{ij}(x, y)$ as $P_i(x)P_j(y)$. Assuming that $L\psi = Q$ is the given equation, $\psi$ is approximated by $\psi_h$ and over $\hat{\Omega}$, cell moments of $\psi_h(x, y) \in S$ are defined as follows

$$\psi_C^{ij} = \int_{-1}^{+1} \int_{-1}^{+1} \frac{P_{ij}(x, y)\psi_h(x, y)\, dxdy}{N_i N_j}. \tag{2.2}$$

Edge moments are moreover given by

$$\psi_E^i = \int_{-1}^{+1} \frac{P_i(s_E)\psi_h(x_E, y_E)ds_E}{N_i} \tag{2.3}$$

where $E$ is a generic symbol corresponding to $L$, $R$, $B$, and $T$ for the left, right, bottom, and top edges respectively, $x_E$ or $y_E$ is $\pm 1$ depending on the particular edge considered, the other coordinate being $s_E$, the coordinate along that edge.

$S$ is a space of functions, which are *polynomials* in this paper. To describe them in a systematic way, let us introduce the spaces of polynomials of degree $i$ in $x$ and $j$ in $y$, $\mathcal{Q}_{ij}(x, y) \equiv \{x^a y^b | 0 \le a \le i, 0 \le b \le j\}$, with in particular $\mathcal{Q}_i \equiv \mathcal{Q}_{ii}(x, y)$ and also the spaces of polynomials of degree $i$ in *x and y*, $\mathcal{P}_i(x, y) \equiv \{x^a y^b | 0 \le a + b \le i\}$, where $a$ and $b$ are integers. For each nodal finite element, we shall call $N_p = card(D)$ the total number of parameters and $N_u$ the number of unknowns which is less than $N_p$ in the weakly discontinuous case, as the interpolation parameters on the left and bottom edges are taken from the neighboring cells or given by the boundary conditions. In the strongly discontinuous case, there are no left and bottom parameters and we have $N_p = N_u$.

In the following, each particular method will be assigned a symbol consisting of two capital letters, *WD* in the weakly discontinuous case and *SD* in the strongly discontinuous one, indexed by the two numbers $N_p$ and $N_u$ in the first case, and by $N_p$ or $N_u$ indifferently in the second case. In the weakly discontinuous case, $(N_p - N_u)/2$ is the number of edge moments conserved between adjacent cells. In most practical cases, this number is one or two.

In both cases, we have programmed all the methods from two to eight unknowns per cell and applied them to multiplicative and nonmultiplicative benchmark problems of the nuclear literature. The discretized equations were obtained using Maple and the algebraic systems were programmed in Fortran for each one of the nodal schemes.

In the weakly discontinuous case, we have given in Hennart et al. [4] a constructive algorithm to deduce the space of functions $S$ if the set of degrees of freedom $D$ is known. In that paper, we always assumed that we had the same number of edge moments on each pair of opposite edges. This is clearly not true in the strongly discontinuous case and we had to adapt the earlier algorithm to that situation for applications in neutron transport problems.