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Positive solutions for a fourth order equation with nonlinear boundary conditions

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Abstract

Existence of positive solutions for a fourth order equation with nonlinear boundary conditions, which models deformations of beams on elastic supports, is considered using fixed points theorems in cones of ordered Banach spaces. Iterative and numerical solutions are also considered.

MSC: 34B18; 34B60; 65L10Nonlocal equation; Fixed point; Positive cone; Kirchhoff beam

1. Introduction

We consider positive solutions to the problem

$$u''''(x) - M\left(\int_0^L u'^2(s)ds\right)u''(x) = f(x, u(x), u'(x)), \quad 0 < x < L, \tag{1}$$

with the boundary conditions

$$u(0) = u''(0) = 0, \tag{2}$$

$$u(L) = 0, \quad u''(L) = g(u'(L)), \tag{3}$$

where $f \in C([0, L] \times \mathbb{R} \times \mathbb{R})$, $g \in C(\mathbb{R})$ and $M \in C(\mathbb{R}^+)$ are real functions.

This problem models the bending equilibrium of an extensible beam (of length L) which is simply supported at $x = 0$ and attached to a fixed nonlinear torsional spring at $x = L$. The force exerted by the foundation is represented by f, and the force due to the spring is given by the function g. The mathematical modeling for elastic beams with nonlocal terms of the type $M(\int u'^2)u''$ can be found in [4,16]. Fourth order equations' modeling beams with nonlinear boundary conditions can be found, for instance, in [2,3,5,6,11,12,14].

The objective of the paper is to prove the existence of positive solutions to (1)–(3) by using a fixed point theorem in cones of positive functions. The main result (Theorem 3) generalizes an early study presented in [15], where linear boundary conditions were considered with $g = 0$. We also discuss the existence of iterative solutions and present some numerical simulations. In this direction, the paper also extends early studies in [13].

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2. Integral representation

In order to obtain a simple integral representation of the problem (1) and (3), we reduce it to a coupled second order system

$$\begin{cases} u'' = M(||u'||_2^2)u + v, \\ u(0) = u(L) = 0, \end{cases} \tag{4}$$

and

$$\begin{cases} v'' = f(x, u, u'), \\ v(0) = 0, \quad v(L) = g(u'(L)), \end{cases} \tag{5}$$

where $|| \cdot ||_p$ denotes $L^p(0, L)$ norms. Let G be the Green's function defined by

$$G(x, t) = \begin{cases} t(L - x)L^{-1} & \text{if} \quad 0 \le t \le x \le L, \\ x(L - t)L^{-1} & \text{if} \quad 0 \le x \le t \le L. \end{cases} \tag{6}$$

Then for each function $h \in C[0, L]$, $w(x) = \int_0^L G(x, t)h(t)dt$ is a solution of the Dirichlet problem $-w'' = h$ with $w(0) = w(L) = 0$. So we conclude from (4) and (5) that

$$u(x) = \int_0^L -G(x, t)\,(\,M(||u'||_2^2)u(t) + v(t)\,)\,dt,$$

where

$$v(t) = \int_0^L -G(t, s)f(s, u(s), u'(s))ds + \frac{t}{L}g(u'(L)).$$

This shows that u is a solution of the problem (1)–(3) if and only if it is a fixed point of the operator T defined by

$$Tu(x) = \int_0^L G(x, t)z(t)dt, \tag{7}$$

where

$$z(t) = \int_0^L G(t, s)f(s, u(s), u'(s))ds - M(||u'||_2^2)u(t) - \frac{t}{L}g(u'(L)). \tag{8}$$

Our existence result is obtained from the Krasnosel'skii fixed point theorem for the compression of a cone (see [10,7]).

Theorem 1. (**Krasnosel'skii**) *Let P be a cone in a Banach space E and let Ω_1 and Ω_2 be open subsets of E, with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. Suppose that*

$$T : P \cap (\Omega_2 \setminus \bar{\Omega}_1) \to P$$

is a completely continuous operator such that

$$||Tu|| \ge ||u|| \quad \text{if} \quad u \in P \cap \partial\Omega_1 \quad \text{and} \quad ||Tu|| \le ||u|| \quad \text{if} \quad u \in P \cap \partial\Omega_2. \tag{9}$$

Then T has a fixed point in $P \cap (\Omega_2 \setminus \bar{\Omega}_1)$.

We apply Krasnosels'kii theorem in a cone of positive functions in the space of continuously differentiable functions $C^1[0, L]$, equipped with the norm

$$||u||_{C^1} = \max\{||u||_\infty, ||u'||_\infty\},$$

where $||w||_\infty = \max_{t \in [0,L]} |w(t)|$. Following [15], we consider the cone

$$P = \{u \in C^1[0, L] \mid u(0) = u(L) = 0 \text{ and } u \text{ is concave}\}, \tag{10}$$

which is a closed set in $C^1[0, L]$.