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Original Article

Statistical moments for solutions of non-linear scalar equations with random Riemann data

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Abstract

In this paper we generalize the solution of random Riemann problem for a scalar equation, for flux function with one inflection point. We introduce both a bifurcation theory for the state space (u_L, u_R) and an efficient numerical method. The statistical moments are obtained from a computable integral exact form. We present some numerical results, considering an uniform distribution, and a bivariate normal distribution. We obtain very good results compared with the solution obtained with Monte Carlo.
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Keywords: Random Riemann problems; Hyperbolic scalar equations; Statistical moments

1. Introduction

In this work, we generalize the ideas proposed in paper [3], where the authors obtain results for the Burgers' equation. Here, we consider the non-linear scalar equation:

$$u_t(x,t) + g(u(x,t))_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \tag{1}$$

with random Riemann data:

$$u(x,0) = \begin{cases} u_L, & \text{if} x < 0, \\ u_R, & \text{if} x > 0. \end{cases} \tag{2}$$

The mathematical formulation is based on some topological concepts. Let $(\Omega, \mathcal{F})$ be a measurable space, with Ω denoting the set of all elementary events and $\mathcal{F}$ a σ-algebra of all possible events in our probability model. For each event $\omega \in \Omega$ we assume that $u(x,t,\omega) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{Q} \subset \mathbb{R}$ and the function $g = g(u(x,t,\omega)) : \mathbb{Q} \to \mathbb{R}$ is $\mathcal{C}^1(\mathbb{Q}, \mathbb{R})$ and has only one inflection point. A very important application of this method developed in this paper is the random Riemann problem for the well known *Buckley–Leverett equation*, see [1], [8], [9]. This equation is very important in the oil engineering, because it describes the flow of two different fluid, for example oil and water, or oil and gas, into a saturated porous medium. The flux function exhibits an inflection point and satisfies the conditions given by (5).

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As in [3] and in [7], we notice that the randomness appears only in the initial data, i.e., u_L and u_R are random variables in $\mathbb{Q}$. We assume that the function g is known. This is a very reasonable hypothesis for many problems. Generically, the function g is given by fundamental laws of mass, momentum and energy balance, besides, fundamental thermodynamical laws, such the first and second laws, see [11].

Uncertainty is observed in several kind of problems and phenomena appearing in biology, physics, engineering, finance, social sciences, and so on, see [15]. It appears naturally in experimental problems due to many errors in measurements, and in many cases this uncertainty can not be avoid or diminished, in such a way that is impossible to determine, *a priori*, the correct initial data, but only some certain statistical quantities of interest like the mean, variance, higher moments and in some cases, a distribution of these data depending on certain physical, chemical, biological or social behaviors. In many cases also, the initial data are not available or it is impossible to perform experiments or measurements to obtain an adequate set of data, then the randomness is important to understand possible behaviors of the solution.

There are several works dealing with random initial Cauchy data. In [10], authors consider a scalar hyperbolic conservation law in $d \geq 1$ spatial dimension, they prove the existence and uniqueness of a random-entropy solution and present a class of numerical schemes of multi-level Monte Carlo finite volume (MLMC-FVM). In [2], authors apply polynomial chaos (PC) methods to study the steady state of an isentropic flow in a dual-throat nozzle with equal throat areas. From numerical experiments they find the shock location. An interesting remark is that when the variance of the initial condition is small the probability density function of the shock location is computed with high accuracy. On the other hand, many terms are needed in the PC expansion to produce reasonable results due to the slow convergence, caused by non-smoothness in random space. We point out that in the proposed method we do not impose conditions on the variance of the initial random Riemann data. In [13], authors use (PC) methods applied to Galerkin projection to study the Burgers' equation with both initial and boundary random conditions. Similar techniques of PC method and kinetic theory are also used by [14] for the Burgers' scalar equation and Euler system with random initial data. In [6], the author used Wiener chaos expansion to separate random and deterministic effects to calculate the statistical moments for the Burgers' equation for a Cauchy problem, where the random initial condition is an explicit function of the spatial variable and the normal random variable with zero mean and unit variance.

In the works before cited there are several numerical experiences on numerical methods, however, they lack of analytical solutions to deal with the problem with random initial data. In this paper, we describe a method to give semi-analytical solution, however, we do not study the problem with general random Cauchy data. Even for scalar equations with deterministic data, the analytical solutions are obtained for a particular initial data, the called Riemann data, see [4]. For the deterministic hyperbolic system, under certain hypothesis, Glimm proved, see [5], that is possible to obtain the solution of a Cauchy problem, by using several Riemann problems, moreover the Riemann problems are used to verify the accuracy of numerical methods. Therefore, the analysis of problems with random Riemann data is very important for theoretical and numerical purposes.

To perform our study, in this work we assume that the distribution of the initial data (2) and their joint probability density function $f_{u_L u_R}$ are known and we obtain an integral form for the statistical moments. These moments are very important for understanding the complete behavior of the phenomena described by the conservation law (1) with the distribution of probability $f_{u_L u_R}$.

In [3], the authors calculated the statistical moments for the Burgers' equation with Riemann random data; for each fixed (x^*, t^*) in the (x, t) space, they used the self-similarity of Riemann solutions in the ray $x = \beta t$ to obtain an explicit integral form for the statistical moments. In [7], author obtains a systematic theory dealing with the Riemann random solution for systems of equations with genuinely nonlinear fields.

Here, we extend the ideas of [3] for a scalar conservation law with one inflection point. We obtain a bifurcation structure for the state space (u_L, u_R) continuously depending on the parameter $\beta = x/t$. This bifurcation structure allows us to represent the statistical moments as a computable integral exact form, similar to that obtained by Cunha in [3]. This integral form gives us condition to obtain a numerical method even more efficient than Monte Carlo to solve the Riemann problem with random initial data.

In Section 2, we describe the random Riemann problem and also the mathematical aspects of the new method to solve random Riemann problems, including the bifurcation structure for (u_L, u_R) and the statistical moments in an integral form. In Section 3 we describe briefly the implementation of the method, and also discuss some aspects concerning efficiency and precision. In Section 4 we present some numerical results for uniform distribution and bivariate normal

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