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Numerical solution of flow problems by stabilized finite element method and verification of its accuracy using a posteriori error estimates ☆

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Abstract

2D flow of incompressible viscous fluid with higher Reynolds number is studied. Galerkin least squares technique of stabilization of the finite element method is investigated and its modification is described. A number of numerical results is presented. Properties of stabilization are discussed. Most important part is the study of the accuracy of the stabilized solution by means of a posteriori error estimates.

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1. Introduction

Many papers have been devoted to the problem of stabilizing the finite element method (FEM) in flow problems. Beside getting ideas about stabilization from other papers, works of Franca, Hughes, and their collaborators [4–6,8,9] provided the theoretical basis for the presented research.

In [2] we modified the Galerkin least squares (GLS) method introduced in [9] and using the modified GLS (we call it semiGLS) we were able to solve flow with markably higher Reynolds number than without stabilization.

In this paper, beside giving some aditional results of semiGLS method, we concentrate mainly on the aspect of accuracy of the stabilization technique.

2. Model problem

Let Ω be an open bounded domain in $\mathbb{R}^2$ filled with an incompressible viscous fluid, and Γ its boundary. We study flow governed by Navier–Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega \times [0, T] \tag{1}$$

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$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \times [0, T] \tag{2}$$

$$\mathbf{u} = \mathbf{g} \text{ on } \Gamma_g \times [0, T] \tag{3}$$

$$-\nu(\nabla \mathbf{u})\mathbf{n} + p\mathbf{n} = \mathbf{0} \text{ on } \Gamma_h \times [0, T] \tag{4}$$

$$\mathbf{u} = \mathbf{u}_0 \text{ in } \Omega,\ t = 0 \tag{5}$$

where t denotes time variable, $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ denotes the vector of flow velocity, p denotes the pressure divided by the density, ν denotes the kinematic viscosity of the fluid supposed to be constant, $\mathbf{f}$ denotes the density of volume forces per mass unit, Γ_g and Γ_h are two subsets of Γ satisfying $\bar{\Gamma} = \bar{\Gamma}_g \cup \bar{\Gamma}_h$, $\mu_{\mathbb{R}^1}(\Gamma_g \cap \Gamma_h) = 0$, $\mathbf{n}$ denotes an outward normal vector to the boundary Γ of unit length, $\mathbf{g}$ is a given function satisfying $\int_\Gamma \mathbf{g} \cdot \mathbf{n}\, \mathrm{d}\Gamma = 0$ in the case of $\Gamma = \Gamma_g$, $\mathbf{u}_0$ is a given flow field satisfying $\nabla \cdot \mathbf{u}_0 = 0$.

3. Approximation of the model problem by FEM

First we recall the weak form of the Navier–Stokes equations (1)–(5), as mixed method (cf. [7]). We define vector function spaces V_g and V by

$$V_g = \{\mathbf{v} = (v_1, v_2)^{\mathrm{T}} | \mathbf{v} \in [H^1(\Omega)]^2; \mathbf{Tr}\, v_i = g_i, i = 1, 2\}, \qquad V = \{\mathbf{v} = (v_1, v_2)^{\mathrm{T}} | \mathbf{v} \in [H_0^1(\Omega)]^2\}$$

where $H^1(\Omega)$, $H_0^1(\Omega)$ and $L_2(\Omega)/\mathbb{R}$ are usual function spaces.

The *weak unsteady Navier–Stokes problem* consists of finding the velocity $\mathbf{u}(t) = (u_1(t), u_2(t))^{\mathrm{T}} \in V_g$ and pressure $p(t) \in L_2(\Omega)/\mathbb{R}$ satisfying for any $t \in [0, T]$:

$$\int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v}\, \mathrm{d}\Omega + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, \mathrm{d}\Omega + \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}\, \mathrm{d}\Omega - \int_\Omega p \nabla \cdot \mathbf{v}\, \mathrm{d}\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}\Omega \tag{6}$$

$$\int_\Omega \psi \nabla \cdot \mathbf{u}\, \mathrm{d}\Omega = 0 \tag{7}$$

$$\mathbf{u} - \mathbf{u}_g \in V \tag{8}$$

for all $\mathbf{v} \in V$ and $\psi \in L^2(\Omega)$, where $\mathbf{u}_g \in V_g$ represents the Dirichlet boundary condition $\mathbf{g}$ in (3), and where we denote

$$\nabla \mathbf{u} : \nabla \mathbf{v} = \frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial v_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial v_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2}.$$

Let us divide the domain Ω into N elements T_K of a triangulation $\mathcal{T}_h$ of shape regular family, such that $\bigcup_{K=1}^N \bar{T}_K = \bar{\Omega}$, $\mu_{\mathbb{R}^2}(T_K \cap T_L) = 0$, $K \neq L$. Let h_K denote the diameter of the element T_K. We consider Hood–Taylor finite elements P_2P_1 and/or Q_2Q_1, which satisfy Babuška–Brezzi stability condition (cf. [1]). Their application leads to the approximation $\mathbf{u}_h \in V_{gh}$ and $p_h \in Q_h$ where

$$V_{gh} = \{\mathbf{v}_h = (v_{h_1}, v_{h_2})^{\mathrm{T}} \in [\mathcal{C}(\bar{\Omega})]^2; v_{h_i}|_{T_K} \in R_2(\bar{T}_K), K = 1, \ldots, N, i = 1, 2, \mathbf{v}_h = \mathbf{g} \text{ in nodes on } \Gamma_g\},$$

$$Q_h = \{\psi_h \in \mathcal{C}(\bar{\Omega}); \psi_h|_{T_K} \in R_1(\bar{T}_K), K = 1, \ldots, N\}$$

where $R_m(\bar{T}_K) = P_m(\bar{T}_K)$, if T_K is a triangle or $Q_m(\bar{T}_K)$, if T_K is a quadrilateral and $\mathcal{C}(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$. We introduce the space

$$V_h = \{\mathbf{v}_h = (v_{h_1}, v_{h_2})^{\mathrm{T}} \in [\mathcal{C}(\bar{\Omega})]^2; v_{h_i}|_{T_K} \in R_2(\bar{T}_K), K = 1, \ldots, N, i = 1, 2, \mathbf{v}_h = \mathbf{0} \text{ in nodes on } \Gamma_g\}$$

Since these function spaces satisfy $V_{gh} \subset V_g$, $V_h \subset V$, and $Q_h \subset L_2(\Omega)/\mathbb{R}$, we can introduce *semidiscrete unsteady Navier–Stokes problem*:

Find $\mathbf{u}_h(t) \in V_{gh}$ and $p_h(t) \in Q_h$, $t \in [0, T]$ satisfying for any $t \in [0, T]$:

$$\int_\Omega \frac{\partial \mathbf{u}_h}{\partial t} \cdot \mathbf{v}_h\, \mathrm{d}\Omega + \int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h\, \mathrm{d}\Omega + \nu \int_\Omega \nabla \mathbf{u}_h : \nabla \mathbf{v}_h\, \mathrm{d}\Omega - \int_\Omega p_h \nabla \cdot \mathbf{v}_h\, \mathrm{d}\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h\, \mathrm{d}\Omega, \quad \forall \mathbf{v}_h \in V_h \tag{9}$$