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Numerical integration in the DGFEM for nonlinear convection-diffusion problems

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Abstract

The effect of numerical integration in the DGFEM for nonlinear convection-diffusion problems in 2D is studied. The volume and line integrals in the space semidiscretization are evaluated by numerical quadratures. The main goal is to estimate the error caused by the numerical integration and to show what numerical quadratures should be used in order to preserve the accuracy of the method with exact integration.

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1. Introduction

In this paper we solve a nonlinear nonstationary convection-diffusion problem in 2D by applying the discontinuous Galerkin finite element method (DGFEM). The DGFEM is based on the combination of ideas and techniques of the finite volume (FV) and finite element (FE) methods. Like the standard FEM this method is based on a piecewise polynomial approximation of the sought solution, but the requirement of the conforming properties is omitted here. Similarly as in the FV method, a numerical flux is used for the approximation of convective terms. (For a survey of various DGFE techniques see, e.g. [1,2].) In practical computations performed by the DGFEM, integrals are evaluated with the aid of quadrature formulae. In this paper we investigate the effect of the numerical integration and show how to choose the integration formulae.

2. Continuous problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and $T > 0$. We are concerned with the following problem: find $u : Q_T = \Omega \times (0, T) \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \sum_{l=1}^{2} \frac{\partial f_l(u)}{\partial x_l} = \varepsilon \Delta u + g \quad \text{in } Q_T, \qquad u|_{\Gamma_D \times (0,T)} = u_D, \qquad \varepsilon \frac{\partial u}{\partial n}\bigg|_{\Gamma_N \times (0,T)} = g_N, \qquad u(.,0) = u^0. \quad (1)$$

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Here the diffusion coefficient $\varepsilon > 0$ is a given constant, f_l $(l = 1, 2)$ are prescribed convective fluxes and g, u_D, g_N and u^0 are given functions.

3. Discrete problem

Let $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ be a system of partitions of $\overline{\Omega}$ into a finite number of closed triangles K with mutually disjoint interiors. We call $\mathcal{T}_h$ triangulations of Ω, but *do not* require the usual conforming properties from the FEM.

We set $h_K = \mathrm{diam}(K)$ and $h = \max_{K\in\mathcal{T}_h} h_K$. By $|K|$ and ρ_K we denote the area of K and the radius of the largest circle inscribed into K, respectively. All elements of $\mathcal{T}_h$ will be numbered in such a way that $\mathcal{T}_h = \{K_i\}_{i\in I}$, where $I \subset \mathbb{Z}^+$. If two elements K_i, $K_j \in \mathcal{T}_h$ contain a nonempty common open part of their sides, we put $\Gamma_{ij} = \Gamma_{ji} = \partial K_i \cap \partial K_j$. For $i \in I$ we set $s(i) = \{j \in I; K_j \text{ is a neighbour of } K_i\}$. The boundary $\partial\Omega$ is formed by a finite number of faces of elements K_i adjacent to $\partial\Omega$. We denote all these boundary faces by S_j, where $j \in I_b \subset \mathbb{Z}^-$, and set $\gamma(i) = \{j \in I_b; S_j \text{ is a face of } K_i\}$, $\Gamma_{ij} = S_j$ for $K_i \in \mathcal{T}_h$ such that $S_j \subset \partial K_i$, $j \in I_b$. Now, writing $S(i) = s(i) \cup \gamma(i)$, we have $\partial K_i = \bigcup_{j\in S(i)} \Gamma_{ij}$, $\partial K_i \cap \partial\Omega = \bigcup_{j\in\gamma(i)} \Gamma_{ij}$. For $i \in I$, by $\gamma_D(i)$ and $\gamma_N(i)$ we denote the subsets of $\gamma(i)$ formed by such indexes j that the faces Γ_{ij} form the parts Γ_D and Γ_N, respectively, of $\partial\Omega$.

Furthermore, we denote by $\mathbf{n}_{ij}$ the unit outer normal to ∂K_i on the face Γ_{ij}, $|\Gamma_{ij}|$ the length of Γ_{ij} and we set $s_h = \{\Gamma_{ij}; j \in S(i), i \in I\}$.

We suppose that the system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ is *regular*, i.e. $h_K/\rho_K \le C_R$ for all $K \in \mathcal{T}_h, h \in (0, h_0)$, and that $h_{K_i} \le C_D|\Gamma_{ij}|$ for all $i \in I$, $j \in S(i)$, $h \in (0, h_0)$.

Over the triangulations $\mathcal{T}_h$ we define for $k \in \mathbb{N}$, $k \ge 1$, the *broken Sobolev spaces*:

$$H^k(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^k(K)\,\forall\, K \in \mathcal{T}_h\}$$

with seminorms defined by $|v|_{H^k(\Omega,\mathcal{T}_h)} = \left(\sum_{K\in\mathcal{T}_h} |v|^2_{H^k(K)}\right)^{1/2}$. For $v \in H^1(\Omega, \mathcal{T}_h)$ we denote the *traces, average* and *jump of the traces* of v on $\Gamma_{ij} = \Gamma_{ji}$ by

$$v|_{\Gamma_{ij}} = \text{trace of } v|_{K_i} \text{ on } \Gamma_{ij}, \qquad v|_{\Gamma_{ji}} = \text{trace of } v|_{K_j} \text{ on } \Gamma_{ji},$$

$$\langle v\rangle_{\Gamma_{ij}} = \tfrac{1}{2}(v|_{\Gamma_{ij}} + v|_{\Gamma_{ji}}) \quad \text{and} \quad [v]_{\Gamma_{ij}} = v|_{\Gamma_{ij}} - v|_{\Gamma_{ji}}.$$

If $j \in \gamma(i)$, then we put $v|_{\Gamma_{ji}} = v|_{\Gamma_{ij}} = \text{trace of } v|_{K_i}$ on Γ_{ij}.

The approximate solution of our problem is sought in the space of discontinuous piecewise polynomial functions $S_h = S^{p,-1}(\Omega, \mathcal{T}_h) = \{v; v|_K \in P^p(K)\,\forall\, K \in \mathcal{T}_h\}$, where $P^p(K)\,(p \ge 1)$ denotes the space of all polynomials on K of degree $\le p$.

In order to introduce the space semidiscretization of problem (1) over the mesh $\mathcal{T}_h$ by the DGFEM, we define the following forms for functions $u, \varphi \in H^2(\Omega, \mathcal{T}_h)$ (the weight σ is defined by $\sigma|_{\Gamma_{ij}} = |\Gamma_{ij}|^{-1}$):

$$(u, \varphi) = \int_\Omega u\varphi\,\mathrm{d}x, \qquad \tilde{a}_h(u, \varphi) = \sum_{i\in I} \int_{K_i} \varepsilon\nabla u \cdot \nabla\varphi\,\mathrm{d}x$$

$$-\sum_{i\in I}\Bigg(\sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \varepsilon\langle\nabla u\rangle \cdot \mathbf{n}_{ij}[\varphi]\,\mathrm{d}S - \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \varepsilon\langle\nabla\varphi\rangle \cdot \mathbf{n}_{ij}[u]\,\mathrm{d}S\Bigg)$$

$$-\sum_{i\in I}\left(\sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \varepsilon\nabla u \cdot \mathbf{n}_{ij}\varphi\,\mathrm{d}S - \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \varepsilon\nabla\varphi \cdot \mathbf{n}_{ij} u\,\mathrm{d}S\right),$$

$$\tilde{J}_h^\sigma(u, \varphi) = \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \sigma[u][\varphi]\mathrm{d}S + \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sigma\, u\, \varphi\,\mathrm{d}S,$$