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Original Article

A dual weighted residual method for an optimal control problem of laser surface hardening of steel

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Abstract

The main focus of this article is on analyzing and implementing a dual weighted residual (DWR) approach for an optimal control problem of laser surface hardening of steel. The problem which is governed by a dynamical system consisting of a semi-linear parabolic equation and an ordinary differential equation is discretized using the finite element method. *A posteriori* error estimates are derived and an adaptive algorithm is formulated. The numerical experiments justify the theoretical results obtained.

MSC: 49J20; 65M60; 65K10

Keywords: Laser surface hardening of steel problem; Adaptive finite element methods; Dual weighted residual method; *A posteriori* error estimates

1. Introduction

In this paper, we develop *a posteriori* error estimates using the *dual weighted approach* for an optimal control problem describing the laser surface hardening of steel. In applications like cutting tools, wheels, driving axles, gears, etc., the surface is stressed. The purpose of laser surface hardening is to increase the hardness of the boundary layer of a workpiece by rapid heating and then a quenching (see Fig. 1). Steel, an alloy, upon heat treatment undergoes a phase-transition to form austenite. The heated zone is quenched by self-cooling, and this leads to the desired hardening effect due to a change in micro-structure. The mathematical model for the laser surface hardening of steel has been studied in [11] and [8]. For an extensive survey on mathematical models for laser material treatments, we refer to [13]. In this article, we consider the Leblond–Devaux model [11].

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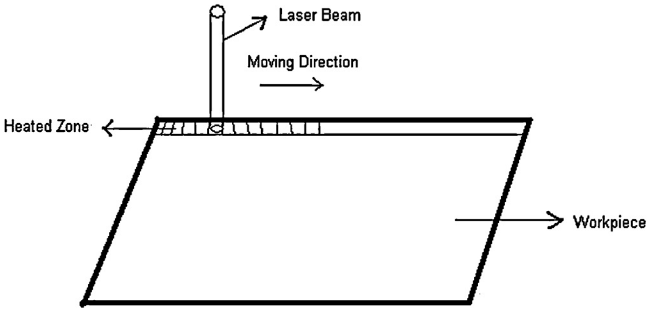

Fig. 1. Laser hardening process.

Let $\Omega \subset \mathbb{R}^2$, denoting the workpiece, be a convex, bounded domain with piecewise Lipschitz continuous boundary $\partial\Omega$ and $I=(0, T)$, $T< \infty$. The evolution of volume fraction of austenite $a(t)$ for a given temperature evolution $\theta(t)$ is described by the following initial value problem:

$$\partial_t a = f(\theta, a) = \frac{1}{\tau(\theta)}[a_{eq}(\theta) - a]_+ \quad in\ \Omega \times I, \tag{1.1}$$

$$a(0) = 0 \quad \text{in}\ \Omega, \tag{1.2}$$

where $a_{eq}(\theta(t))$, denoted as $a_{eq}(\theta)$ for notational convenience, is the equilibrium volume fraction of austenite and τ depends only on the temperature θ. The term $[a_{eq}(\theta) - a]_+ = (a_{eq}(\theta) - a)\mathcal{H}(a_{eq}(\theta) - a)$ denotes the non-negative part of $a_{eq}(\theta) - a$, where $\mathcal{H}$ is the Heaviside function defined by

$$\mathcal{H}(s) = \begin{cases} 1 & s > 0 \\ 0 & s \leq 0. \end{cases}$$

For theoretical and computational reasons, $[a_{eq}(\theta) - a]_+$ is approximated with the help of the regularized Heaviside function given by

$$\mathcal{H}_\epsilon(s) = \begin{cases} 1 & s \geq \epsilon \\ 10\left(\frac{s}{\epsilon}\right)^6 - 24\left(\frac{s}{\epsilon}\right)^5 + 15\left(\frac{s}{\epsilon}\right)^4 & 0 < s \leq \epsilon \\ 0 & s \leq 0 \end{cases}$$

as $[a_{eq}(\theta) - a]_+ \approx (a_{eq}(\theta) - a)\mathcal{H}_\epsilon(a_{eq}(\theta) - a)$ and the resulting problem, which is the regularized version of the original problem (1.1) and (1.2) (see [1,8]) is considered throughout in the article.

Using the Fourier law of heat conduction, the temperature evolution can be obtained by solving the non-linear energy balance equation given by

$$\rho c_p \partial_t \theta - \mathcal{K}\Delta\theta = -\rho L a_t + \alpha u \quad in\ \Omega \times I, \tag{1.3}$$

$$\theta(0) = \theta_0 \quad \text{in}\ \Omega, \tag{1.4}$$

$$\nabla\theta.\mathbf{n} = 0 \quad \text{on}\ \partial\Omega \times I, \tag{1.5}$$

where the density ρ, the heat capacity c_p, the thermal conductivity $\mathcal{K}$ and the latent heat L are assumed to be positive constants. The term $u(t)\alpha(x, t)$ describes the volumetric heat source due to laser radiation, $u(t)$ being the time dependent control variable. Since the main cooling effect is the self cooling of the workpiece, homogeneous Neumann condition is assumed on the boundary. Also, θ_0 denotes the initial temperature.