Available online at www.sciencedirect.com

ScienceDirect

Mathematics and Computers in Simulation 103 (2014) 39–59

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

Original article

Basins of attraction for several optimal fourth order methods for multiple roots

Beny Neta [a,*], Changbum Chun [b,1]

[a] *Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA 93943, United States*
[b] *Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea*

Received 28 February 2014; received in revised form 14 March 2014; accepted 31 March 2014
Available online 12 April 2014

Abstract

There are very few optimal fourth order methods for solving nonlinear algebraic equations having roots of multiplicity m. Here we compare five such methods, two of which require the evaluation of the $(m-1)$st root. The methods are usually compared by evaluating the computational efficiency and the efficiency index. In this paper all the methods have the same efficiency, since they are of the same order and use the same information. Frequently, comparisons of the various schemes are based on the number of iterations required for convergence, number of function evaluations, and/or amount of CPU time. If a particular algorithm does not converge or if it converges to a different solution, then that particular algorithm is thought to be inferior to the others. The primary flaw in this type of comparison is that the starting point represents only one of an infinite number of other choices. Here we use the basin of attraction idea to recommend the best fourth order method. The basin of attraction is a method to visually comprehend how an algorithm behaves as a function of the various starting points.
Published by Elsevier B.V. on behalf of IMACS.

MSC: 65H05; 65B99

Keywords: Iterative methods; Order of convergence; Rational maps; Basin of attraction; Julia sets; Conjugacy classes

1. Introduction

There is a vast literature on the solution of nonlinear equations, see for example Ostrowski [21], Traub [27], Neta [13], Petković et al. [23] and references therein. In the recent book by Petković et al. [23], they have shown that some methods are a rediscovery of old ones and some are just special cases of other methods. Most of the algorithms are for finding a simple root α of a nonlinear equation $f(x)=0$. In this paper we are interested in the case that the root is of a known multiplicity $m>1$. Clearly, one can use the quotient $f(x)/f'(x)$ which has a simple root where $f(x)$ has a multiple

* Corresponding author. Tel.: +1 831 656 2235; fax: +1 831 656 2355.
E-mail addresses: bneta@nps.edu, byneta@gmail.com (B. Neta), cbchun@skku.edu (C. Chun).
[1] Tel.: +82 31 299 4523; fax: +82 31 290 7033.

http://dx.doi.org/10.1016/j.matcom.2014.03.007
0378-4754/Published by Elsevier B.V. on behalf of IMACS.

root. Such an idea will not require a knowledge of the multiplicity, but on the other hand will require higher derivatives. For example, Newton's method for the function $F(x) = f(x)/f'(x)$ will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - [f(x_n)f''(x_n)/f'(x_n)]}. \tag{1}$$

If we define the efficiency index of a method of order, p as

$$I = p^{1/d}, \tag{2}$$

where d is the number of function- (and derivative-) evaluation per step then this method has an efficiency of $2^{1/3} = 1.2599$ instead of $\sqrt{2} = 1.4142$ for Newton's method for simple roots.

There are very few methods for multiple roots when the multiplicity is known. These method are based on the function $G(x) = \sqrt[m]{f(x)}$ which obviously has a simple root at α, the multiple root with multiplicity m of $f(x)$. The first one is due to Schröder [24] and it is also referred to as modified Newton,

$$x_{n+1} = x_n - m\, u_n, \tag{3}$$

where

$$u_n = \frac{f(x_n)}{f'(x_n)}. \tag{4}$$

Another method based on the same G is Laguerre's-like method

$$x_{n+1} = x_n - \frac{\lambda u_n}{1 + sgn(\lambda - m)\sqrt{((\lambda - m)/m)[(\lambda - 1) - \lambda u_n f''(x_n)/f'(x_n)]}} \tag{5}$$

where λ ($\neq 0$, m) is a real parameter. When $f(x)$ is a polynomial of degree n, this method with $\lambda = n$ is the ordinary Laguerre method for multiple roots, see Bodewig [5]. This method converges cubically. Some special cases are Euler–Cauchy, Halley, Ostrowski and Hansen–Patrick [9]. Petković et al. [22] have shown the equivalence between Laguerre family (5) and Hansen–Patrick family. When $\lambda \to m$ the method becomes second order given by (3). Two other cubically convergent methods that sometimes mistaken as members of Laguerre's family are: Euler–Chebyshev [27] and Osada's method [20].

Other methods for multiple roots can be found in [30,14,28,7,8]. Li et al. [11] have developed 6 fourth order methods based on the results of Neta and Johnson [15] and Neta [16]. We will give just those two that are optimal. A method is called optimal if it attains the order 2^n and uses $n+1$ function-evaluations. Thus a fourth-order optimal method is one that requires 3 function- and derivative-evaluation per step.

In the next section we will present the five optimal fourth-order methods to be analyzed. In the two sections following it, we will analyze the basins of attraction to compare all these fourth order optimal methods for multiple roots. The idea of using basins of attraction was initiated by Stewart [26] and followed by the works of Amat et al. [1–4], Scott et al. [25] and Chun et al. [6].

Neta et al. [19] and Neta and Chun [18] have compared several methods for multiple roots but they have not considered the methods appearing here.

2. Optimal fourth order methods for multiple roots

There are very few methods of optimal order for multiple roots. Li et al. [11] have developed six different methods but only two are optimal, in the sense of Kung and Traub [10]. These are denoted here by LCN5 and LCN6. Liu and Zhou [12] have developed two optimal fourth order methods, denoted here by LZ11 and LZ12. We also discuss a family of methods developed Zhou et al. [31].

- LCN5 (Li et al. [11])

$$\begin{aligned} y_n &= x_n - \frac{2m}{m+2} u_n, \\ x_{n+1} &= x_n - a_3 \frac{f(x_n)}{f'(y_n)} - \frac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)}, \end{aligned} \tag{6}$$