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An iterative approach to the solution of an inverse problem in linear elasticity

A. Ellabib [a], A. Nachaoui [b,*]

[a] *Université Cadi Ayyad, Faculté des Sciences et Techniques, Département de Mathématiques et Informatique, Avenue Abdelkrim Elkhattabi, BP 549, Guéliz Marrakech, Morocco*
[b] *Laboratoire de Mathématiques Jean Leray, UMR 6629, Université de Nantes/CNRS/ECN, 2 rue de la Houssinière, BP 92208, 44322 Nantes, France*

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Abstract

This paper presents an iterative alternating algorithm for solving an inverse problem in linear elasticity. A relaxation procedure is developed in order to increase the rate of convergence of the algorithm and two selection criteria for the variable relaxation factors are provided. The boundary element method is used in order to implement numerically the constructing algorithm. We discuss this implementation, mention the use of Krylov methods to solve the obtained linear algebraic systems of equations and investigate the convergence and the stability when the data is perturbed by noise.

Keywords: Boundary elements; Inverse problem; Elasticity equations; LU decomposition; Iterative methods.

1. Introduction

A vast body of engineering experience shows that the theory of linear elasticity allows an accurate modeling of many natural or manufactured solid materials (civil engineering structures, transportation vehicles, machines, the Earth's mantle and rocks mechanics [9]) and provides an essential tool for analysis and design.

When the governing system of partial differential equations, i.e. the equilibrium, constitutive and kinematics equations, have to be solved with the appropriate initial and boundary conditions for the displacement and/or traction vectors, i.e. Dirichlet, Neumann or mixed boundary conditions the associated problems are called direct problems and their existence and uniqueness have been well established. When one or more of the conditions for solving the direct problem are partially or entirely unknown then an inverse problem may be formulated to determine the unknowns from specified or measured system responses.

The main type of inverse problems that arise in the context of linear elasticity, and more generally of the mechanics of deformable solids, are similar to those encountered in other areas of physics involving continuous media and distributed physical quantities, e.g., acoustics, electrostatics and electromagnetism. They are usually motivated by the desire or need to overcome a lack of information concerning the properties of the system (a deformable solid body or structure). It should be noted that most of the inverse problems are ill posed and hence they are more difficult to solve than the direct problems. It is well known that they are generally instable, i.e. the existence, uniqueness and stability of their

* Corresponding author.
E-mail addresses: ellabib@fstg-marrakech.ac.ma (A. Ellabib), nachaoui@math.univ-nantes.fr (A. Nachaoui).

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solutions are not always guaranteed, see e.g. Hadamard [4]. Identification of inaccessible boundary values (Cauchy problem in elasticity) is a classical example of inverse problem. This inverse problem, in which both displacement and traction boundary conditions are prescribed only on a part of the boundary of the solution domain whilst no information is available on the remaining part of the boundary, can be encountered in many situations [5,16].

Recently, an approximate solution to the Cauchy problem for Poison equation has been determined by one of the authors [7,11], using an alternating iterative method which reduced the problem to solving a sequence of well-posed boundary value problems. Our goal in this paper is to extend this algorithm in conjunction with the boundary element method (BEM) to the Cauchy problem in elasticity.

The paper is organized as follows. In the next section, we present the direct and an inverse problem for linear elasticity. Then, we give an iterative approach for the Cauchy problem for linear elasticity equation and also we expose two automatic selection of the relaxation factor. We describe in Section 4 boundary element method for elasticity equations. In Section 5, the technique of implementation of this iterative approach are detailed. Numerical results are presented in Section 6 which explore the convergence and stability of algorithm and also we compare some linear iterative method.

2. Mathematical model

2.1. Direct problem statement

The mathematical formulation of the 2D elasticity problem in the case of an isotropic linear elastic material which occupies an open bounded domain $\Omega \subset \mathbb{R}^2$ with boundary Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1, \Gamma_2 \neq \emptyset$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$ is described as follows.

Let $w = (u, v)^T$ be the displacement vector and b the volume force vector. Here $(.,.)^T$ denotes the transpose of a vector or a matrix. Let us define the matrices:

$$\mathcal{D} = \begin{pmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{pmatrix}, \qquad \mathcal{E} = \begin{pmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{2\partial y} & \dfrac{\partial}{2\partial x} \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{pmatrix} \tag{1}$$

Then the strain vector $\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})^T$ is given by

$$\varepsilon = \mathcal{E}w \tag{2}$$

The strain tensor ε is related to the stress vector $\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T$ by the constitutive law:

$$\sigma = \frac{2G}{1-2\nu}\mathcal{C}\varepsilon \tag{3}$$

where G and ν are respectively the Shear modulus and Poisson ratio.

The equilibrium equations are given by

$$\mathcal{D}\sigma = b \tag{4}$$

If we now substitute the constitutive law (3) into the equilibrium equation (4), and use the kinematic relations (2) of the elasticity tensor for an isotropic linear elastic material, we obtain the following Lamé system or the Navier equations:

$$G\Delta u + \frac{G}{1-2\nu}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) = b_1 \text{ in } \Omega \qquad G\Delta v + \frac{G}{1-2\nu}\left(\frac{\partial^2 u}{\partial x}\partial y + \frac{\partial^2 v}{\partial y^2}\right) = b_2 \text{ in } \Omega \tag{5}$$

The solution of Eqs. (5) must satisfy prescribed boundary conditions on the boundary Γ of the body, which are based either on the displacements u and v, or the boundary traction t and s. The boundary conditions can be written into the following types:

$$u(X) = \tilde{u}(X), \qquad v(X) = \tilde{v}(X) \quad \text{for } X \in \Gamma_1 \tag{6}$$