Available online at www.sciencedirect.com

ScienceDirect

Mathematics and Computers in Simulation 77 (2008) 218–227

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

On Chebyshev-type discrete quasi-interpolants

María José Ibáñez-Pérez

Departamento de Matemática Aplicada, Universidad de Granada, Campus Universitario de Fuentenueva s/n, 18071 Granada, Spain

Available online 31 August 2007

Abstract

In this paper new discrete quasi-interpolants on the real line are defined with good error constants for enough regular functions. Some oversampling is permitted in order to have some freedom degrees and so a minimization problem is established. This problem has always a solution that can be characterized in terms of the best uniform approximation by constant functions to some appropriate splines. Some examples are given and the error is analyzed.

PACS: 41A05; 41A15; 65D05; 65D07

Keywords: B-splines; Discrete quasi-interpolants; Quasi-interpolation error; Chebyshev-type discrete quasi-interpolants

1. Introduction

Spline quasi-interpolation is a local method for constructing spline approximants to a function or discrete data without solving any linear or non-linear systems, so it is a very interesting tool in practice as well as in establishing theoretical results in spline analysis (see [7,8,6,9,14]).

A univariate spline discrete quasi-interpolant (dQI) of a given function f on the uniform partition $h\mathbb{Z}$ with spacing h takes values $f(jh)$, $j\in\mathbb{Z}$, to define an approximant $Q_h f$ of f which is a B-spline series $\sum_{i\in\mathbb{Z}}\lambda_i(f(h\cdot))M(\dot{h}-i)$, where M is a B-spline on the uniform partition $\mathbb{Z}$ of the real line induced, and λ_i is a linear functional defined by $\lambda_i(g):=\sum_{j\in J}\gamma_j g(i-j)$ for some finite subset $J\subset\mathbb{Z}$ and real numbers γ_j. When $J=\{0\}$ the Schoenberg's quasi-interpolation operator S_h is obtained. It is given by

$$S_h f := \sum_{i\in\mathbb{Z}} f(hi)M(\dot{h}-i),$$

and it reproduces the space $\mathbb{P}_1$ of polynomials of the first degree. Other choices of J are needed in order to achieve the exactness of the dQI on the space $\mathbb{P}(M)$ of the polynomials of maximal degree included in the space spanned by the integer translates of M or in all the spline space (cf. e.g. [6,11]). As we are interested in the first case, let J be such a subset. Since $||f-Q_h f||_\infty \le (1+||Q||_\infty)\mathrm{dist}(f,\mathbb{P}(M))$, with $Q=Q_1$ (see e.g. [9, p. 144]), it is quite natural to construct a Q_h of a small infinity norm, thus a small constant for the quasi-interpolation error is obtained. This has been done in [1–3,10], but that construction does not take into account the class of the function to be approximated, so another method is required in order to define dQIs with good error constants. In this paper, we permit again some oversampling, that is a J with $\#J\ge\#(\mathbb{Z}\cap\mathrm{supp}M)$ and $\mathbb{Z}\cap\mathrm{supp}M\subset J$, and we consider a more adequate expression for the quasi-interpolation error. In Section 2, we motivate the construction of such dQIs and we recall the linear

E-mail address: mibanez@ugr.es.

0378-4754/$32.00
doi:10.1016/j.matcom.2007.08.018

constraints on the coefficients of the linear forms to achieve the maximal exactness. In Section 3 the Chebyshev-type dQIs are defined and characterized, and the existence of such dQIs is established. Finally, in Section 4 some examples are given, and the corresponding errors are considered in Section 5.

2. Preliminaries

Let $M := M_n$ be the B-spline of order $n \geq 3$, with support $[-n/2, n/2]$ (cf. e.g. [13,14]). Let us consider a dQI $Q := Q_n = Q_{n,J}$

$$Qf = \sum_{i \in \mathbb{Z}} \left(\sum_{j \in J} \gamma_j f(i-j) \right) M(\cdot - i), \tag{1}$$

for some $\gamma_j \in \mathbb{R}$, $j \in J$, and its scaled quasi-interpolation operator Q_h, $h > 0$. We recall that $\mathbb{P}(M)$ is the space $\mathbb{P}_{n-1}$ of polynomials of degree at most $n-1$. Let us suppose that Q (and then Q_h) is exact on $\mathbb{P}_{n-1}$. Taking into account that $n = 2\nu$ or $2\nu - 1$ for some $\nu \geq 2$, can be verified that, for $x \in [kh, (k+1)h)$, $k \in \mathbb{Z}$, $Q_h f(x)$ is a sum of the values $f((k - \max J - \nu + 1)h), \ldots, f((k - \min J + \nu)h)$. Therefore, the quasi-interpolation error $f(x) - Q_h f(x)$ is null on $\mathbb{P}_{n-1}$, and, for $f \in C^n(\mathbb{R})$, the Peano's Theorem (see e.g. [12, chapter 4]) gives

$$f(x) - Q_h f(x) = \frac{1}{(n-1)!} \int_{(k - \max J - \nu + 1)h}^{(k - \min J + \nu)h} K_h(x,t) f^{(n)}(t)\, \mathrm{d}t,$$

where $K_h(x,t) = (x-t)_+^{n-1} - Q_h[(\cdot - t)_+^{n-1}](x)$. Writing $x = (k+\xi)h$ for some $\xi \in [0, 1)$, we get

$$f(x) - Q_h f(x) = \frac{h^n}{(n-1)!} \int_{-\max J - \nu + 1}^{-\min J + \nu} K(\xi, \tau) f^{(n)}(h(k+\tau))\, \mathrm{d}\tau,$$

where

$$K(\xi, \tau) = (\xi - \tau)_+^{n-1} - Q[(\cdot - \tau)_+^{n-1}](\xi).$$

Hence

$$|f(x) - Q_h f(x)| \leq \frac{h^n}{(n-1)!} ||f^{(n)}||_{\infty, I_{h,J,k}} \int_{-\max J - \nu + 1}^{-\min J + \nu} |K(\xi, \tau)|\, \mathrm{d}\tau,$$

with $I_{h,J,k} := [(k - \max J - \nu + 1)h, (k - \min J + \nu)h]$.

In order to obtain a quasi-interpolation error as small as possible, we construct Q to be a good approximation of the Peano's kernel $K(\xi, \tau)$. Denote $e_l(\xi) := \xi^l$, $l \in \mathbb{N} \cup \{0\}$. For each fixed $\tau \in [-\max J - \nu + 1, -\min J + \nu]$, let $p^*(\xi) := \sum_{l=0}^{n+r} a_l(\tau) e_l(\xi)$ be the best uniform approximation by polynomials of degree at most $n+r$, $r \geq 0$ to $(\cdot - \tau)_+^{n-1}$ on $[0, 1]$, i.e.

$$||p^* - (\cdot - \tau)_+^{n-1}||_{\infty,[0,1]} = \inf_{p \in \mathbb{P}_{n+r}} \max_{\xi \in [0,1]} |p(\xi) - (\xi - \tau)_+^{n-1}|.$$

Since Q is exact on $\mathbb{P}_{n-1}$, we get $Qp^* = \sum_{l=0}^{n-1} a_l(\tau) e_l + \sum_{l=n}^{n+r} a_l(\tau) Q e_l$, whence

$$p^* - Qp^* = \sum_{l=n}^{n+r} a_l(\tau)(e_l - Qe_l).$$

Thus, $\sum_{l=n}^{n+r} |a_l(\tau)|\, ||e_l - Qe_l||_{\infty,[0,1]}$ approximates $|K(\cdot, \tau)|$ and we proposed to construct Q by minimizing the errors $||e_l - Qe_l||_{\infty,[0,1]}$, $n \leq l \leq n+r$. When $r = 0$ only the error $e_n - Qe_n$ appears in the above expression for the approximation of $|K(\cdot, \tau)|$, and so Q is determined by how well it approximates the first non-preserved monomial, e_n. Therefore, we have a Chebyshev-type construction (and a Chebyshev-type result is obtained).

Since the exactness is a main property to be imposed to Q, we recall how the monomials are expressed as linear combinations of the integer translates of M (see [4,5]).