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On a new characterization of finite jump discontinuities and its application to vertical fault detection

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Abstract

The purpose of this paper is to give a new characterization of the finite jump discontinuities of multivariate functions, in terms of the divergence of sequences related to the gradients of discrete least-squares polynomial approximations of the function. We also consider how this result can be applied to vertical fault detection.

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1. Introduction

The approximation of discontinuous functions has become an active field of research, due to its applications in computer graphics, medical imaging, geophysical sciences, etc. Particularly, the reconstruction of a faulted surface from a set of scattered data points is a common problem.

Let f be a real multivariate function that presents finite jump discontinuities on a certain subset $\mathcal{F}$ of an open set $\Omega \subset \mathbb{R}^d$. The first stage in the process of approximation of f is usually to localize the subset $\mathcal{F}$. Then, the function f is reconstructed by using a fitting method. There exists a number of methods for both stages (cf., for example ([1], Chapter IX), [2] and references therein).

In two previous papers (cf. [3,4]; see also [5]), we proposed algorithms for fault detection which were based on the following result: Let $\mathbf{c} \in \Omega$. Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of compact sets with non-zero measure, contained in the closure of Ω, such that $\{\mathbf{c}\} = \cap_{n \in \mathbb{N}} K_n$ and $\lim_{n \to +\infty} \text{meas}\, K_n = 0$, where meas K_n denotes the measure of K_n. Finally, for any $n \in \mathbb{N}$, let $\Pi^l_{K_n} f$ be the least-squares approximation of f in the space $P_l(K_n)$ of polynomial functions of degree $\leq l$ defined on K_n and let

$$J_{K_n}(f) = \frac{1}{\text{meas}\, K_n} \int_{K_n} \|\nabla(\Pi^l_{K_n} f)(\mathbf{x})\|^2 \, \mathrm{d}\mathbf{x},$$

where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^d$. Then, under some additional hypotheses on f, $\mathcal{F}$ and $(K_n)_{n \in \mathbb{N}}$, if f presents at $\mathbf{c}$ a finite jump discontinuity, the sequence $(J_{K_n}(f))_{n \in \mathbb{N}}$ is divergent, whereas, if f is continuous at $\mathbf{c}$, that

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sequence remains bounded. In practice, the function f is only known at a finite set of points in Ω. Hence, it is not possible to compute the polynomials $\Pi^l_{K_n} f$, which have to be replaced by discrete least-squares polynomial approximations. The purpose of this paper is just to prove a similar result in terms of these discrete approximations (cf. [6] for the particular case $d = 2$).

This paper is organized as follows. In Section 2, we shall introduce some preliminary notations and results. Section 3 will be devoted to the new characterization of finite jump discontinuities of a function. This result will serve in Section 4 to provide an heuristic justification of a method for vertical fault detection.

2. Preliminaries

Let $m \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For any square real matrix Λ of order m, $\det \Lambda$ will denote the determinant of Λ and, for any $\mathbf{u} \in \mathbb{R}^m$ and for $j = 1, \ldots, m$, $\Lambda(j, \mathbf{u}^{\mathrm{T}})$ will denote the matrix obtained from Λ when its jth column is replaced by the column vector $\mathbf{u}^{\mathrm{T}}$ (here the super-index T means transposition).

For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, we shall write $\|\mathbf{u}\|$ and $\mathbf{u} \cdot \mathbf{v}$ for the Euclidean norm of $\mathbf{u}$ and the Euclidean inner product of $\mathbf{u}$ and $\mathbf{v}$, respectively. Likewise, if $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_m)$, we shall use the notations:

$$\bar{\mathbf{u}} = \frac{1}{m}\sum_{i=1}^{m} u_i, \qquad \mathbf{u} \odot \mathbf{v} = (u_1 v_1, \ldots, u_m v_m) \quad \text{and} \quad \sigma_{\mathbf{uv}} = \overline{\mathbf{u} \odot \mathbf{v}} - \bar{\mathbf{u}}\ \bar{\mathbf{v}}.$$

Let $d \in \mathbb{N}^*$. The closure, the boundary and the cardinality of any set $\mathcal{O} \subset \mathbb{R}^d$ will be denoted by $\bar{\mathcal{O}}$, $\partial\mathcal{O}$ and $\operatorname{card}\mathcal{O}$, respectively. Likewise, we shall write $P_1(\mathbb{R}^d)$ for the space of polynomial functions of degree ≤ 1 defined on $\mathbb{R}^d$.

Given an open set $\Omega \subset \mathbb{R}^d$, a finite set $A \subset \Omega$ containing a $P_1(\mathbb{R}^d)$-unisolvent subset and a function $f : \Omega \to \mathbb{R}$, we shall denote by $\Pi_A f$ the discrete least-squares polynomial of degree ≤ 1 that fits the data set $\{(\mathbf{a}, f(\mathbf{a})) | \mathbf{a} \in A\}$, i.e. the unique solution of the problem:

$$\Pi_A f \in P_1(\mathbb{R}^d) \quad \text{and} \quad \Pi_A f = \arg \min_{p \in P_1(\mathbb{R}^d)} \sum_{\mathbf{a} \in A} (f(\mathbf{a}) - p(\mathbf{a}))^2. \tag{2.1}$$

Likewise, we shall write

$$J_A(f) = \frac{1}{\operatorname{meas} K} \int_K \|\nabla(\Pi_A f)(\mathbf{x})\|^2 \, d\mathbf{x}, \tag{2.2}$$

where K is any compact set, with non-empty interior, which contains A. Since $\nabla(\Pi_A f)$ is equal on $\mathbb{R}^d$ to a constant vector, say $\boldsymbol{\alpha}$, $J_A(f)$ is, in fact, the number $\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}$, independent of K.

3. A new characterization of finite jump discontinuities

3.1. Hypotheses

Let $d \in \mathbb{N}^*$. Let Ω be an open subset of $\mathbb{R}^d$ and let $\mathcal{F}$ be a subset of Ω satisfying the following assumption:

$$\text{there exists an open set } \omega_1 \subset \Omega \text{ with a Lipschitz-continuous boundary such that } \mathcal{F} \text{ is an open subset of } \partial\omega_1 \text{ in the trace topology of } \partial\omega_1. \tag{3.1}$$

From now on, we shall write $\omega_2 = \Omega \setminus \bar{\omega}_1$. Likewise, let $f : \Omega \to \mathbb{R}$ be a function such that

$$f \text{ is continuous on } \Omega \setminus \mathcal{F} \text{ and Lipschitz-continuous on } \omega_1 \text{ and } \omega_2 \tag{3.2}$$

and

$$\forall \mathbf{x} \in \mathcal{F}, \; f(\mathbf{x}) = f_1(\mathbf{x}) \neq f_2(\mathbf{x}), \tag{3.3}$$

where, for $i = 1$ and 2, the function f_i is the continuous extension of $f|_{\omega_i}$ to $\bar{\omega}_i$. We remark that the existence of f_i is a simple consequence of (3.2). In fact, f_i is a Lipschitz-continuous function on $\bar{\omega}_i$. For any $\mathbf{x} \in \mathcal{F}$, an arbitrary value can be assigned to $f(\mathbf{x})$. For convenience, we have assumed in (3.3) that $f = f_1$ on $\mathcal{F}$.