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# A recursive procedure to obtain a class of orthogonal polynomial wavelets

M. Moncayo [a,*], R.J. Yáñez [b]

[a] *Department of Applied Mathematics and Statistics, Technical University of Cartagena (UPCT), E-30202 Cartagena, Spain*
[b] *Department of Applied Mathematics and Institute Carlos I for Theoretical and Computational Physics, University of Granada, E-18071 Granada, Spain*

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**Abstract**

In this paper we present a recursive approach to generate complex orthogonal polynomial systems. The systems belong to a class of polynomial wavelets successfully introduced by Skopina [M. Skopina, Orthogonal polynomial Shauder bases in $\mathcal{C}[-1, 1]$ with optimal growth of degrees, Sb. Math. 192 (3) (2001) 433–454; M. Skopina, Multiresolution analysis of periodic functions, East J. Approx. 3 (1997) 203–224]. Consequently, by using the obtained recursive-type relation, it is possible to generate a great variety of complex polynomial functions which satisfy useful wavelet-like properties. We prove some additional multiscale results concerning these systems. More precisely, we state a practical two-scale relation and the decomposition and reconstruction formulae which determine the multiresolution analysis framework. From the reconstruction formula, we obtain the recursive approach which provides the Skopina's systems. Finally, a numerical example in which explicit complex orthogonal polynomials are found recursively is presented.


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## 1. Introduction

Orthogonal wavelets and frames have been successfully applied in a variety of engineering fields. Hence, the construction of new types of wavelets, including polynomial systems [3–5,9–11], is still a growing relevant area of research. The classical wavelet theory due to Mallat and Meyer is developed in the framework of a multiresolution analysis (MRA) scheme. By combining scaling and wavelet functions associated with the classical multiresolution scheme together with functions defined by orthogonal polynomials with respect to a weight function $\omega(x)$, Skopina [9] introduced a class of complex valued orthogonal polynomials which inherit the multiscale properties from the standard scaling and wavelet functions. Furthermore, if the classical wavelet functions are sufficiently localized and smooth then, the complex polynomial systems constitute an orthogonal Schauder basis of optimal degree for the space of continuous functions and define a MRA scheme in $L^2_\omega(a, b)$ (see [9], Theorems 1 and 2, respectively). This motivates that they may be thought as a class of polynomial wavelets with interesting properties. This paper is devoted to complete the MRA properties established by Skopina. More precisely, in Section 2 we present elements of the basic theory of wavelets and we review the Skopina's results. Section 3 contains the proof of the two-scale relations which provide the

* Corresponding author.
*E-mail addresses:* maria.moncayo@upct.es (M. Moncayo), ryanez@ugr.es (R.J. Yáñez).

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decomposition–reconstruction formulae. From the reconstruction formula, we derived the recursive procedure to generate the systems. Section 4 is devoted to establish a practical conjugation relation which reduces by half the complexity of computations and we also state a result concerning the zero moments property of the systems under consideration. Finally, an explicit example of the derived recurrence-type relation by using the Laguerre polynomials is given.

## 2. Background on wavelets

As starting point we consider a MRA scheme (see [1,2,6,7]). This is a nested sequence of closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$, such that

(i) $\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots \subset L^2(\mathbf{R})$.
(ii) $\overline{\bigcup_j V_j} = L^2(\mathbf{R}), \bigcap_{j\in\mathbf{Z}} V_j = \{0\}$.
(iii) $f \in V_j \Leftrightarrow f(2^j\cdot) \in V_0$.
(iv) There exists a $\varphi \in V_0$, called the scaling function, such that the system $\{\varphi(\cdot - n); n \in \mathbf{Z}\}$ is an orthonormal basis for $V_0$ and it is also assumed the condition:

$$\left|\int_{\mathbf{R}} \varphi(x)\,\mathrm{d}x\right| = 1.$$

An important property of the scaling function $\varphi$ is the *two-scale relation*:

$$\varphi(x) = \sqrt{2}\sum_n h_n \varphi(2x - n), \tag{1}$$

where $(h_n) \in l^2(\mathbf{Z})$. By taking the Fourier transform of both sides, we obtain $\hat{\varphi}(\xi) = m_0(\xi/2)\hat{\varphi}(\xi/2)$, where $m_0$ is the $2\pi$-periodic function given by $m_0(\xi) = 1/\sqrt{2}\sum_n h_n\,\mathrm{e}^{-\mathrm{i}n\xi}$. The scaling function is used to construct the associated wavelet function, $\psi$. It must be chosen such that $\{\psi(x - n)\}$ is an orthonormal basis of the space $W_0$, the orthogonal complement of $V_0$ in $V_{-1}$. Then $V_0 \oplus W_0 = V_{-1}$. If such a $\psi(x)$ can be found then (see [6], p. 236):

$$\{\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x - k); k \in \mathbf{Z}\},$$

is an orthonormal basis of $W_j$, the orthogonal complement of $V_j$ in $V_{j-1}$ and $\{\psi_{j,k}(x)\}_{j,k\in\mathbf{Z}}$ is an orthonormal basis of $L^2(\mathbf{R})$. A relation between the scaling and the wavelet function is given by

$$\psi(x) = \sqrt{2}\sum_n g_n \varphi(2x - n), \tag{2}$$

where $g_n = (-1)^n \bar{h}_{1-n}$. We point out that if $\hat{\varphi}$ takes only real values, then $m_0(\xi)$ is also a real function, and

$$\hat{\psi}(\xi) = \mathrm{e}^{\mathrm{i}(\xi/2)}\,\rho(\xi), \tag{3}$$

where $\rho(\xi) = \overline{m_0(\xi/2 + \pi)}\hat{\varphi}(\xi/2)$ is also real. There exist some orthogonality relations between the scaling and the wavelet functions. More precisely, $\langle \varphi_{j,k}, \varphi_{j,k'}\rangle = \delta_{k,k'}$, $\langle \psi_{j,k}, \psi_{j,k'}\rangle = \delta_{k,k'}$, and $\langle \varphi_{j,k}, \psi_{j',k'}\rangle = 0$.

The systems:

$$\sum_{l\in\mathbf{Z}} \hat{\varphi}(\xi + 2\pi l)\,\mathrm{e}^{\mathrm{i}l\omega}, \qquad \sum_{l\in\mathbf{Z}} \hat{\varphi}_{j,k}(\xi + 2\pi l)\,\mathrm{e}^{\mathrm{i}l\omega} \quad \text{and} \quad \sum_{l\in\mathbf{Z}} \hat{\psi}_{j,k}(\xi + 2\pi l)\,\mathrm{e}^{\mathrm{i}l\omega},$$

are orthonormal in $L^2[0, 2\pi]$ and make possible to introduce a periodic MRA [2,9,10]. Two prototypes of wavelet bases are given by the Shannon system and its generalization, the Meyer system (see [2], p. 137). Both systems constitute particular cases of a full class of Meyer-type wavelets (see [13], p. 50). More precisely, the scaling functions are given by

$$\hat{\varphi}(\xi) = \sqrt{\int_{\xi-\pi}^{\xi+\pi} \gamma(x)\,\mathrm{d}x}, \tag{4}$$