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A convergence result for a least-squares method using Schauder bases

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Abstract

In this work we introduce a method, by using the least-squares method and a Schauder basis, which provides a numerical solution for a wide class of linear differential or integral equations. In addition, we give a convergence result and an application.

Keywords: Least-squares; Schauder basis; Differential equation; Integral equation

1. Introduction

A lot of linear differential or integral equations can be stated in terms of bounded linear operators between functions spaces. In such a formulation, the solution is the preimage of a known function. In this paper, we determine a numerical approximation of the solution, making use of the properties of a Schauder basis in a Banach space and the least-squares method.

Let us start by posing the problem. Let X and Y be Banach spaces (the scalar field $\mathbb{K}$ will be the real or the complex one), let $D : X \to Y$ be a bounded, linear and one-to-one operator from X onto Y, and let $y_0 \in Y$. The question is:

$$\text{find } x_0 \in X \text{ such that } Dx_0 = y_0. \quad \text{(P)}$$

Let us recall that a sequence $\{x_n\}_{n\geq 1}$ in a Banach space X is called a *Schauder basis* provided that for each x in X there are unique scalars $\{a_n\}_{n\geq 1}$ such that

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

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The scalars $a_n \in \mathbb{K}$ are the *coefficients* of x in the basis $\{x_n\}_{n\geq 1}$. If $x \in X$ admits the above expression and $n \geq 1$, we define $P_n x$ by the element in X

$$P_n x := \sum_{k=1}^{n} a_k x_k.$$

It is a well-known fact that the operator $P_n : X \to X$ is a bounded linear operator on X and the sequence $\{P_n\}_{n\geq 1}$ is called the *sequence of projections* associated with the basis $\{x_n\}_{n\geq 1}$.

We denote by $\langle x_1, \dots, x_n\rangle$ the linear span of $\{x_1, \dots, x_n\}$. Let $I = [a, b]$ a real interval. Given $p \in \mathbb{R}$, $m, k, d \in \mathbb{N}$ with $1 \leq p < +\infty$, $m \geq 0$, $k \geq 0$ and $d \geq 1$, $L_p(I^d)$ stands the Banach space of p-integrable functions on I^d, $C^k(I^d)$ denotes de Banach space of k times continuously differentiable functions, and $W_p^m(I^d)$ is the usual Sobolev space.

We recall that $L_2(I^d)$ and $W_2^m(I^d)$ are Hilbert spaces endowed with their usual inner products. For $p = 2$, we denote $W_2^m(I^d)$ by $H^m(I^d)$. Finally, $\mathbb{P}_m(I)$ is the linear space of restrictions on I of all real polynomials of degree $\leq m$.

It is straightforward to give bases for the sequence spaces c_0 or ℓ_p $(1 \leq p < \infty)$ (see ref. [5]) and it is clear that a basis for a Hilbert space is a Schauder basis of it. For bases in the functions spaces $L_p[a, b]$ for $1 \leq p < \infty$, $C^k([0, 1]^d)$ or $W_p^m([0, 1]^d)$, we refer to refs. [3,4,11].

In ref. [10] the inverse image of an element by means of a one-to-one bounded and linear operator is obtained making use of an adequate version of the best approximation theorem for Banach spaces and some properties of Schauder bases.

In ref. [9] a Schauder basis $\{y_n\}_{n\geq 1}$ in Y is considered. The solution for the problem is obtained by using a direct method with a low computational cost. However, it has a restriction: we need an explicit expression for $D^{-1}(y_n)$, that is, we must solve the problem in the case that the load function y_0 be y_n. In certain non-restrictive cases, $D^{-1}y_n$ can be calculated, for instance, by means of the Tau method [8,7].

In this paper, we show another method for solving the same problem. Under certain assumptions, a Schauder basis $\{x_n\}_{n\geq 1}$ in X gives a Schauder basis $\{Dx_n\}_{n\geq 1}$ for Y. Then we calculate the best approximation of y_0 in $\{Dx_1, \dots, Dx_n\}$ by a least-squares method. From this approximation we can determine a function in X that can be considered as an approximation of x_0. Under suitable conditions, we prove the main result of this paper, which guarantees the convergence of the method. Thus, we establish the converge of a least-squares method to solve the (P) problem. A review of some methods of least-squares, their applications and convergence results can be viewed in ref. [2].

2. Analytic results

The next analytic theorem is our key result for the applications:

Theorem 2.1. *Let $(X, \|\cdot\|)$ and $(Y, |\cdot|)$ be Banach spaces such that Y is endowed with an inner product, whose associated norm $\|\cdot\|_2$ satisfies*

$$\text{there exists } k > 0 \text{ such that for all } y \in Y, \quad \|y\|_2 \leq k|y|. \tag{1}$$

Let us assume that $D : X \to Y$ is a linear and one-to-one operator from X onto Y, so that $D^{-1} : (Y, \|\cdot\|_2) \to (X, \|\cdot\|)$ is bounded. Suppose in addition that $\{x_n\}_{n\geq 1}$ is a Schauder basis in X, x_0 is an element in X, and that for all $n \geq 1$,

$$\sum_{k=1}^{n} \beta_k^{(n)} Dx_k$$

is the orthogonal projection of Dx_0 onto $\langle Dx_1, \dots, Dx_n\rangle$. Then,

$$\lim_{n\to\infty} \left| x_0 - \sum_{k=1}^{n} \beta_k^{(n)} x_k \right| = 0.$$

Proof. The fact that the bijective linear operator $D^{-1} : (Y, \|\cdot\|_2) \to (X, \|\cdot\|)$ is continuous, the inequality (1) and the open mapping theorem guarantee that the operator $D : (X, \|\cdot\|) \to (Y, |\cdot|)$ is an isomorphism from X onto Y. Hence, the sequence $\{Dx_n\}_{n\geq 1}$ is a Schauder basis in $(Y, |\cdot|)$ and as a consequence, the subspace spanned by it is