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Efficient spectral ultraspherical-dual-Petrov–Galerkin algorithms for the direct solution of $(2n+1)$th-order linear differential equations

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Abstract

Some efficient and accurate algorithms based on ultraspherical-dual-Petrov–Galerkin method are developed and implemented for solving $(2n+1)$th-order linear elliptic differential equations in one variable subject to homogeneous and nonhomogeneous boundary conditions using a spectral discretization. The key idea to the efficiency of our algorithms is to use trial functions satisfying the underlying boundary conditions of the differential equations and the test functions satisfying the dual boundary conditions. The method leads to linear systems with specially structured matrices that can be efficiently inverted. Numerical results are presented to demonstrate the efficiency of our proposed algorithms.

JEL classification: 65M70; 65N35; 35C10; 42C10

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1. Introduction

Spectral methods have developed rapidly in the past four decades. They have been applied successfully to numerical simulations in many fields. They have gained new popularity in automatic computations for a wide class of physical problems in fluid and heat flow. The principal advantage of spectral methods lies in their ability to achieve accurate results with substantially fewer degrees of freedom.

Spectral methods (see, for instance, [3,4,13,18,29]) involve representing the solution to a problem in terms of a truncated series of smooth global functions. They give very accurate approximations for a smooth solution with relatively few degrees of freedom.

This paper aims to develop some efficient spectral algorithms based on ultraspherical-Petrov–Galerkin method (UPGM) for $(2n+1)$th-order elliptic differential equations in one variable. It is worthy noting that ultraspherical polynomials are a special type of Jacobi polynomials.

The majority of books and research papers dealing with the theory of ordinary differential equations, or their practical applications to technology and physics, contain mainly results from the theory of second-order linear differential equations, and some results from the theory of some special linear differential equations of higher even order. However

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there is only a limited body of literature on spectral methods for dispersive, namely, third- and higher odd-order equations. In particular, relatively few studies are devoted to third- and higher odd-order equations in finite intervals. This is partly due to the fact that direct collocation methods for higher odd-order boundary problems lead to condition numbers of high order, typically of order N^{2k}, where N is the number of retained modes and k is the order of equation. This high condition number will lead to instabilities caused by rounding errors (see, for instance, [19,24]).

The study of odd-order equations is of interest, for example, the third-order equation is of fundamental mathematical interest since it lacks symmetry. Also, it is of physical interest since it contains a type of operator which appears in many commonly occurring partial differential equations such as the Kortweg–de Vries equation. Monographs like those of (Swanson [30] and Mckelvey [23]), which include chapters on oscillation properties of third-order differential equations, are exceptional. The interested reader in applications of odd-order differential equations is referred to the monograph by (Gregus [15]), in which many physical and engineering applications of third-order differential equations are discussed (see, pp. 247–258).

From the numerical point of view, Doha and Abd-Elhameed [7], Doha and Bhrawy [8–10], Doha et al. [11] and Shen [26,27] have constructed efficient spectral-Galerkin algorithms using compact combinations of orthogonal polynomials for solving elliptic equations of second-, fourth-, and $2n$ th-order in various situations. Recently, Shen [28] introduced an efficient spectral dual-Petrov–Galerkin algorithm for third- and fifth-order equations using compact combinations of Legendre polynomials. Also some studies are devoted to third- and fifth-order differential equations in finite intervals (see, [21,22,14]).

In this paper we are concerned with the direct solution techniques for $(2n+1)$th-order elliptic equations, using ultraspherical-dual-Petrov–Galerkin approximations. Our algorithms lead to discrete linear systems with specially structured matrices that can be efficiently inverted. The techniques of Shen [28] and Abd-Elhameed [1], and some interesting cases can be obtained directly as special cases from our proposed ultraspherical-Petrov–Gelerkin approximations. This partially motivates our interest for making such a generalization. Another motivation is, when one uses finite difference or finite element methods to solve numerically certain physical problems in unbounded domains [16], one often restricts calculations to some bounded domains and impose certain conditions on artificial boundaries, which destroy the accuracy. If one uses spectral methods associated with ultraspherical polynomials in unbounded domains, then the above-mentioned troubles could be remedied. The last motivation is, for some special problems such as singular differential equations, see [17], the ultraspherical polynomials could be preferable.

The main differential operator in odd-order differential equations is not symmetric, so it is convenient to use a Petrov–Galerkin method. The difference between Galerkin and Petrov–Galerkin methods, is that the test and trial functions in Galerkin method are the same, but for Petrov–Galerkin method, the trial functions are chosen to satisfy the boundary conditions of the differential equation, and the test functions are chosen to satisfy the dual boundary conditions.

We organize the materials of this paper as follows. In Section 2, we give some properties of ultraspherical polynomials. In Section 3, we discuss an algorithm for solving $(2n+1)$th-order elliptic linear differential equations subject to homogeneous boundary conditions. In Section 4, we are concerned with the same equations subject to nonhomogeneous boundary conditions. Numerical results are given in Section 5 to show the efficiency of our algorithms. Some concluding remarks are given in Section 6.

2. Some properties of ultraspherical polynomials

The ultraspherical polynomials (a special type of Jacobi polynomials) associated with the real parameter $(\lambda > -(1/2))$ are a sequence of orthogonal polynomials on the interval $(-1, 1)$, with respect to the weight function $w(x) = (1-x^2)^{\lambda-(1/2)}$, i.e.

$$\int_{-1}^{1} (1-x^2)^{\lambda-(1/2)} C_m^{(\lambda)}(x) C_n^{(\lambda)}(x)\,\mathrm{d}x = \begin{cases} 0, & m \neq n, \\ h_n, & m = n, \end{cases} \tag{1}$$

where

$$h_n = \frac{\sqrt{\pi}\, n!\, \Gamma(\lambda + (1/2))}{(2\lambda)_n (n+\lambda)\Gamma(\lambda)}, \qquad (2\lambda)_n = \frac{\Gamma(n+2\lambda)}{\Gamma(2\lambda)}.$$