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Error analysis for a non-standard class of differential quasi-interpolants[☆]

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Abstract

Given a B-spline M on $\mathbb{R}^s$, $s \geq 1$ we consider a classical discrete quasi-interpolant Q_d written in the form

$$Q_d f = \sum_{i \in \mathbb{Z}^s} f(i) L(\cdot - i),$$

where $L(x) := \sum_{j \in J} c_j M(x-j)$ for some finite subset $J \subset \mathbb{Z}^s$ and $c_j \in \mathbb{R}$. This fundamental function is determined to produce a quasi-interpolation operator exact on the space of polynomials of maximal total degree included in the space spanned by the integer translates of M, say $\mathbb{P}_m$. By replacing $f(i)$ in the expression defining $Q_d f$ by a modified Taylor polynomial of degree r at i, we derive non-standard differential quasi-interpolants $Q_{D,r} f$ of f satisfying the reproduction property

$$Q_{D,r}\, p = p, \text{ for all } p \in \mathbb{P}_{m+r}.$$

We fully analyze the quasi-interpolation error $Q_{D,r} f - f$ for $f \in C^{m+2}(\mathbb{R}^s)$, and we get a two term expression for the error. The leading part of that expression involves a function on the sequence $c := (c_j)_{j \in J}$ defining the discrete and the differential quasi-interpolation operators. It measures how well the non-reproduced monomials are approximated, and then we propose a minimization problem based on this function.

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1. Introduction

Let M be a s-variate B-spline, i.e. a compactly supported non-negative polynomial piecewise function defined on $\mathbb{R}^s$, $s \geq 1$, normalized by $\sum_{i \in \mathbb{Z}^s} M(\cdot - i) = 1$. Let $\mathcal{S}(M) := \text{span}(M(\cdot - i))_{i \in \mathbb{Z}^s}$ be the cardinal spline space spanned by

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the shifts of M. A quasi-interpolant for $\mathcal{S}(M)$ is a linear map into $\mathcal{S}(M)$ which is local, bounded, and reproduces some (nontrivial) polynomial space (see [11], p. 63). The standard structure for a quasi-interpolant is given by the expression

$$Qf := Q(f) := \sum_{i \in \mathbb{Z}^s} \lambda f(\cdot + i) M(\cdot - i), \tag{1}$$

λ being some suitable linear functional.

Standard quasi-interpolants are very useful schemes for approximating multivariate functions. They are defined without solving any linear or non-linear system of equations, and provide the approximation power of $\mathcal{S}(M)$ for smooth functions.

We refer to [11,10,16] for the various methods in the literature for constructing such quasi-interpolants.

In [1–3,5,7,15], the construction of new quasi-interpolation operators Q exact on some space of polynomials $\mathbb{P}_m$ and having small infinity norms is considered (see [1,4,15] for the non-uniform univariate case). Then, an estimate for the quasi-interpolation error $Qf - f$ is obtained taking into account the Lebesgue inequality

$$\|f - Qf\|_\infty \le (1 + \|Q\|_\infty)\, \mathrm{dist}\,(f, \mathbb{P}_m)$$

In [6], new discrete bivariate quasi-interpolants based on box splines are constructed by minimizing an expression involved in a particular estimate of the quasi-interpolation error defined from the errors for some non-reproduced monomials.

In a recent paper [9], we have proposed a general method to construct new differential quasi-interpolation operators from discrete ones in such a way that the new operators reproduce polynomials to the highest possible degree. That method has been used in [8] to define C^1 cubic quasi-interpolating splines on a type-2 triangulation starting from a particular discrete quasi-interpolant.

The aim of this paper is to combine both methods to derive new explicit differential spline quasi-interpolants, based on uniform type triangulation approximating regularly distributed data. They only use the values of the function to be approximated as well as their derivatives up to some prescribed order at the grid points.

In Section 2, we give some notations and recall the construction of the modified differential quasi-interpolants as well as some related results. In Section 3, we establish an integral representation for the error of approximation, from which we estate a minimization problem leading to the non-standard class of differential quasi-interpolant. Finally, we give an example to show that the propose method can be produce good operators.

2. Notations and preliminaries

For a real valued function f and $k \in \mathbb{N}$, we say $f \in C^k(\mathbb{R}^s)$ if f is k times continuously differentiable in the following sense: the directional derivatives of order l, $l = 0, \ldots, k$, at $x \in \mathbb{R}^s$ along the direction $y \in \mathbb{R}^s$ defined as

$$D_y^l f(x) = \frac{d^l}{dt^l} f(x + ty)|_{t=0}$$

exist and depend continuously on x. When the directional derivative exists for y, it can be extended to multiples by defining

$$D_{\alpha y}^l f(x) = \alpha^l D_y^l f(x), \alpha \in \mathbb{R}.$$

For $f \in C^k(\mathbb{R}^s)$, we introduce

$$\left| D^k f \right| = \sup_{x \in \mathbb{R}^s} \sup \left\{ \left| \underset{y}{\overset{k}{D}} f(x) \right| : y \in \mathbb{R}^s, \|y\| = 1 \right\},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^s$ and $|f|_k = \sum_{|l|=k} \left| D^l f \right|$. We use the notations $|\alpha| := \sum_{k=1}^s \alpha_k$ for the length of the multi-integer $\alpha := (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$ with non-negative components, $\alpha! := \prod_{k=1}^s \alpha_k!$, $x^\alpha := \prod_{k=1}^s x_k^{\alpha_k}$, and $m_\alpha(x) := x^\alpha / \alpha!$.