Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 75 (2007) 113–125
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Approximate solution of linear ordinary differential equations with variable coefficients

Xian-Fang Li*

Institute of Mechanics and Sensor Technology, School of Civil Engineering and Architecture, Central South University, Changsha, Hunan 410083, China

Received 11 July 2005; received in revised form 22 September 2006; accepted 27 September 2006
Available online 13 November 2006

Abstract

In this paper, a novel, simple yet efficient method is proposed to approximately solve linear ordinary differential equations (ODEs). Emphasis is put on second-order linear ODEs with variable coefficients. First, the ODE to be solved is transformed to either a Volterra integral equation or a Fredholm integral equation, depending on whether initial or boundary conditions are given. Then using Taylor's expansion, two different approaches based on differentiation and integration methods are employed to reduce the resulting integral equations to a system of linear equations for the unknown and its derivatives. By solving the system, approximate solutions are determined. Moreover, an nth-order approximation is exact for a polynomial of degree less than or equal to n. This method can readily be implemented by symbolic computation. Illustrative examples are given to demonstrate the efficiency and high accuracy of the proposed method.

Keywords: Approximate solution; Taylor's expansion; Ordinary differential equations; Variable coefficient

1. Introduction

Besides numerical approaches, some approximate methods such as the power series expansion method and the perturbation technique are powerful tools for solving initial-value or boundary-value problems associated with ordinary differential equations (ODEs) [2]. The power series expansion method is very suitable for a class of initial value problems. As for the perturbation technique, it pertains only to a class of ODEs with small parameters. Apart from these two methods, some new approaches have been formulated to solve initial-value or boundary-value problems of ODEs, particularly for second-order linear ODEs with variable coefficients. For example, Holubec and Stauffer employed the analytic continuation approach to determine an efficient solution of a linear ODE [6]. Killingbeck [8] suggested a shooting method for solving a special second-order Schrödinger equation. Everitt et al. gave orthogonal polynomial solutions of linear ODEs [5]. In addition, Coutsias et al. dealt with ODEs with rational function coefficients through a spectral method [3]. Nevertheless, all of the above-mentioned methods express the solution to be determined in terms of infinite series, which must be truncated as a sum of finite terms in practical implementation. On the other

* Fax: +86 7318879358.
E-mail address: xfli@mail.csu.edu.cn (X.-F. Li).

doi:10.1016/j.matcom.2006.09.006

hand, by means of Taylor's expansion and the Runge–Kutta approximation, Sarafyan derived a continuous approximate solution of ODEs [12]. For a class of initial-value problems of ODEs with deviating arguments, Allinger and Henry gave a method for finding best polynomial approximations [1]. Based on the differential transfer matrix method, an analytical treatment of linear ODEs has been analyzed by Khorasani and Adibi [7].

In this paper, a novel, simple, efficient approach is proposed to determine approximate solutions of a linear ODE, in particular for a second-order linear ODE. For such an equation under initial or boundary conditions, the equation to be solved is transformed to either a Volterra integral equation or a Fredholm integral equation, depending on appropriate conditions. Then using the Taylor's expansion of the unknown function, the resulting integral equations are further reduced to a system of linear equations for the unknown function and its derivatives through differentiation and integration techniques. Finally, the accuracy of the given nth-order approximate solution is analyzed and several examples are given to illustrate the effectiveness of the present method.

2. ODEs subject to initial conditions

Due to the significance of second-order linear ODEs in practice, in what follows we focus our attention to this class of equations, for simplicity, i.e.

$$\varphi''(x) + p(x)\varphi'(x) + q(x)\varphi(x) = g(x), \tag{1}$$

where the variable coefficients $p(x)$ and $q(x)$ are known functions, satisfying $p(x) \in C^1(I)$, $q(x) \in C(I)$, I being the interval of interest, and the nonhomogeneous term $g(x)$ is a known continuous function, $g(x) \in C(I)$. For higher-order linear ODEs with variable coefficients, the method is completely similar and omitted here.

We start with considering the second-order linear ODE (1) under initial conditions

$$\varphi(x_0) = \alpha_0, \tag{2}$$

$$\varphi'(x_0) = \alpha_1, \tag{3}$$

where $x_0 \in I$, α_0 and α_1 are prescribed constants.

To solve the above equation, we first introduce some notation, namely

$$f_{(j)}(x) = \frac{1}{(j-1)!}\int_{x_0}^{x} (x-t)^{j-1} f(t)\,\mathrm{d}t, \quad j \geq 1, \tag{4}$$

and

$$f_{(0)}(x) = f(x). \tag{5}$$

Furthermore, by using integration by parts, it is easily verified that the following equality

$$\int_{x_0}^{x} (x-t)^j p'(t)\,\mathrm{d}t = j!\,p_{(j)}(x) - p(x_0)\,(x-x_0)^j\,, \quad j \geq 0, \tag{6}$$

holds for $p(x) \in C^1(I)$.

Now we integrate both sides of Eq. (1) with respect to x from x_0 to x. Using the given initial conditions and performing integration by parts, we have

$$\varphi'(x) - \alpha_1 + p(x)\varphi(x) - p(x_0)\alpha_0 + \int_{x_0}^{x} [q(t) - p'(t)]\varphi(t)\,\mathrm{d}t = g_{(1)}(x). \tag{7}$$

Furthermore, we continue to integrate both sides of Eq. (7) with respect to x from x_0 to x, and get

$$\varphi(x) + \int_{x_0}^{x} k_v(x,t)\varphi(t)\,\mathrm{d}t = f(x), \tag{8}$$

where

$$k_v(x,t) = p(t) + (x-t)[q(t) - p'(t)], \tag{9}$$

$$f(x) = g_{(2)}(x) + \alpha_0 + [\alpha_0 p(x_0) + \alpha_1](x - x_0). \tag{10}$$